Solution to Test 2, Math 2568.
Instructions: Closed book and notes. Show reasoning.

1a. Determine A^{-1} if it exists for A given below. If A^{-1} does not exist, explain why not.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution: In order to find B so that $AB = I$, werow-reduce the corresponding augmented matrix $(A|I)$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

and therefore

$$B = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

has the property that $AB = I$ and hence from definition $A^{-1} = B$ as defined above. (To check algebra, we can directly verify that $AB = I$.)

1b. For vectors $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$,

i. Determine projection $proj_{\mathbf{q}}\mathbf{u}$.

ii. Use result in **i.** to express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{q}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{q}$.

Solution: Note $\frac{\mathbf{u}^T\mathbf{q}}{\mathbf{q}^T\mathbf{q}} = \frac{(2)(1)+(1)(1)+(0)(1)}{1^2+1^2+1^2} = \frac{3}{3} = 1$. So,

$$\mathbf{u}_1 = proj_{\mathbf{q}}\mathbf{u} = \left(\frac{\mathbf{u}^T\mathbf{q}}{\mathbf{q}^T\mathbf{q}}\right)\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We notice that this means that $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and we can double check the algebra by verifying $\mathbf{u}_2^T\mathbf{q}_1 = [1, 0, -1]\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ as expected.

2a. Consider the following polynomials equipped with the usual addition and scalar multiplication operations:

$$S = \{p \in P_2, \quad \text{with the product } p(1) \times p(0) = 0\}$$

Determine if S is a vector space. Give clear reasons.

Solution: Let's check property C1. Suppose $p, q \in S$. Then each is a quadratic and satisfy

$$p(0)p(1) = 0 \quad \text{and} \quad q(0)q(1) = 0$$

We need to show that $p + q \in S$. Since sum of quadratic function is a quadratic function, we have $p + q \in P_2$. But we have further restriction in order for $p + q \in S$. We need to check that

$$(p(1) + q(1)) \times (p(0) + q(0)) = 0$$

But left hand side of the above, when expanded out and using above condition on p and q gives

$$p(1)p(0) + q(1)p(0) + q(0)p(1) + q(1)q(0) = q(1)p(0) + q(0)p(1) \ ,$$

which looks like it need not be zero. To confirm that this is true, we choose a counter example. Take $p(x) = 1 - x, q(x) = x \in S$ since $p(1)p(0) = 0$ and $q(1)q(0) = 0$; however, $p(x) + q(x) = 1$ for all x and therefore

$$(p(1) + q(1))\,(p(0) + q(0)) = 1 \neq 0$$

Hence (C1) is not satisfied and S cannot be a vector space.

2b. Take $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, determine if the following set W is a subspace:

$$W = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{a}^T\mathbf{x} = 0 \ \ \textbf{and} \ \ \mathbf{b}^T\mathbf{x} = 0\right\}$$

Solution: As far as (C1), take $\mathbf{x}, \mathbf{y} \in W$; this implies $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and

$$\mathbf{a}^T\mathbf{x} = 0 \ , \mathbf{b}^T\mathbf{x} = 0 \quad , \text{ and } \mathbf{a}^T\mathbf{y} = 0 \ , \mathbf{b}^T\mathbf{y} = 0$$

Now, we know $\mathbf{x} + \mathbf{y} \in \mathbb{R}^3$ since each vector is in vector space $\mathbb{R}^3$. Also, we have

$$\mathbf{a}^T(\mathbf{x} + \mathbf{y}) = \mathbf{a}^T\mathbf{x} + \mathbf{a}^T\mathbf{y} = 0 + 0 = 0 \quad \text{and} \quad \mathbf{b}^T(\mathbf{x} + \mathbf{y}) = \mathbf{b}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} = 0 + 0 = 0$$

Therefore $\mathbf{x} + \mathbf{y} \in W$ and property (C1) is satisfied.

Now take $\mathbf{x} \in W$ implying above condition satisfied for $\mathbf{x}$. Take c a scalar. Then, we need to check if $c\mathbf{x} \in W$. First since $\mathbf{x} \in \mathbb{R}^3$ implies that $c\mathbf{x} \in \mathbb{R}^3$. Now, check

$$\mathbf{a}^T(c\mathbf{x}) = c\mathbf{a}^T\mathbf{x} = c(0) = 0 \quad \text{and} \quad \mathbf{b}^T(c\mathbf{x}) = c\mathbf{b}^T\mathbf{x} = c(0) = 0$$

Hence $c\mathbf{x} \in W$ and property (C2) is verified. Along with (C1) verified earlier, this shows that W is a subspace of $\mathbb{R}^3$.

Just a comment: note each of $\mathbf{a}^T\mathbf{x} = 0$ and $\mathbf{b}^T\mathbf{x} = 0$ represent planes through the origin, if we write out the scalar product of $\mathbf{x} = [x_1, x_2, x_3]^T$ with given vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore, when both conditions are satisfied, we have the intersection of two planes, which results in a straightline through the origin. Since such a straight line is the span of a single vector, and spans of vectors form a subspace, it is not surprising that our result led to a subspace.

3a. For matrix A as below, determine
i. A basis for the null space,
ii. Nullity and rank

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix} \tag{1}$$

Solution Since null space is the set of vectors $\mathbf{X} \in \mathbb{R}^4$ so that $A\mathbf{X} = \mathbf{0}$, it follows that the resulting augmented matrix for the linear system row reduces as below:

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 0 & 2 & 1 & 0 \end{pmatrix} \Rightarrow^{R_3 - R_1} \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow^{\frac{1}{2} R_2} \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

It follows that $x_3 = s$, $x_4 = t$ are arbitrary and that $x_1 = -2s - t$, $x_2 = -s - t$, and so

$$\mathbf{x} = s \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

and so a basis for null space is $\left\{ [-2, -1, 1, 0]^T, [-1, -1, 0, 1]^T \right\}$ since the set of two vectors are not multiple of each other and therefore linearly independent. Since two vectors are involved in the basis, we have nullity equals two. Hence rank of A = 4-2=2.
3b. Determine if the set $S = \left\{ 1 + x, x + x^2, x^2 \right\}$ spans P_2.
Solution: Take arbitrary element in P_2, which is in the form $b_0 + b_1 x + b_2 x^2$ and ask if for some (a_1, a_2, a_3) we have

$$a_1 (1 + x) + a_2 (x + x^2) + a_3 x^2 = b_0 + b_1 x + b_2 x^2$$

Equating each of the coefficients of 1, x and x^2, we get

$$a_1 = b_0 \quad , \quad a_1 + a_2 = b_1 \quad , \quad a_2 + a_3 = b_3$$

This linear system we can solve for a_1, a_2, a_3 through augmented matrix and noticing at the end that there is no restriction on b_1, b_2, b_3. Alternately, since the above system of equation is easily solvable by first substituting $a_1 = b_0$ into the second equation to get $a_2 = b_1 - b_0$ and then substituting into the third equation to get $a_2 = b_3 - (b_1 - b_0)$. Thus, we can solve for (a_1, a_2, a_3) in terms of b_0, b_1, b_2 without any restriction on the $b's$, implying $Span\ S = P_2$.

4a. Suppose $S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

i. Use Gram-Schmidt orthogonalization on S to construct an orthogonal basis for $\mathbb{R}^2$.

ii. How do you determine an orthonormal basis using results in **i.** ?

Solution We have $\mathbf{u}_1 = \mathbf{w}_1 = [1,1]^T$. We need

$$\mathbf{u}_2 = \mathbf{w}_2 - proj_{\mathbf{u}_1}\mathbf{w}_2 = \mathbf{w}_2 - \alpha_{2,1}\mathbf{u}_1 \quad \text{where} \quad \alpha_{2,1} = \frac{\mathbf{w}_2^T \mathbf{u}_1}{\|\mathbf{u}_1\|^2}$$

Calculating $\alpha_{2,1} = \frac{1+2}{1^2+1^2} = \frac{3}{2}$, resulting in $\mathbf{u}_2 = [1,2]^T - \frac{3}{2}[1,1]^T = \left[-\frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2}[-1,1]^T$. Easily checked that $\mathbf{u}_2^T\mathbf{u}_1 = 0$, as it must and therefore $\{\mathbf{u}_1, \mathbf{u}_2\}$ as given above forms orthogonal basis for $\mathbb{R}^2$. To get orthonormal basis, we need to divide by length of each vector, and so we have orthonormal basis given by

$$\left\{ \frac{1}{\sqrt{2}}[1,1]^T, \frac{1}{\sqrt{2}}[-1,1]^T \right\}$$

4b. Prove that $T \to \mathbb{R}^3 \to \mathbb{R}^2$ defined below is a linear transformation and determine its matrix

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 \\ x_2 + x_2 + 2x_3 \end{pmatrix}$$

Solution: We notice that using preoperties of matrix multiplication, we have

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A\mathbf{x}$$

Hence $T(\mathbf{x}) = A\mathbf{x}$ for the 2×3 matrix A as above. From matrix multiplication properites since $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and for scalar α, $A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$, it follows that $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, and $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$, and hence T as defined above is a linear transformation with matrix A.