Maxwell's Equations: Application of Stokes and Gauss' theorem

The object of this write up is to derive the so-called Maxwell's equation in electro-dynamics from laws given in your Physics class. Maxwell's form of electro-dynamic equations are more convenient the resulting Partial Differential Equations (PDE) can be solved in many cases[1] This allows, for instance, calculation of the electro-magnetic wave propagation caused by an oscillating dipole built out of a wire loop carrying alternating current. It also allows calculation of electric and magnetic fields without simplifying assumptions you have see in your physics class.

1. Gauss' Law for Electric Field in differential form

Take a region in space having a smooth charge density $\rho(x, y, z)$. Then, according to Gauss' Law that you learn in Physics, the electric field $\mathbf{E}$ satisfies:

$$\iint_S \mathbf{E} \cdot \mathbf{dS} = \frac{q}{\epsilon_0} \tag{1}$$

where q is the charge enclosed within the region with closed boundary S. Since charge is distributed all over the region, with density ρ, the total charge within S is

$$q = \iiint_{\mathcal{E}} \rho \, dV, \tag{2}$$

where $\mathcal{E}$ is the solid region inside S. Therefore, equation (1) becomes,

$$\iint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \iiint_{\mathcal{E}} \rho \, dV \tag{3}$$

However, since there are no point charges inside $\mathcal{E}$ by assumption, $\mathbf{E}$ has continuous partial derivatives and we obtain from divergence Theorem and (3):

$$\iiint_{\mathcal{E}} div \, \mathbf{E} dV = \iint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \iiint_{\mathcal{E}} \rho \, dV \tag{4}$$

Therefore, for any region $\mathcal{E}$, it follows that

$$\iiint_{\mathcal{E}} \left\{ div \, \mathbf{E} - \frac{\rho}{\epsilon_0} \right\} dV = 0$$

We divide the expression above by V, the volume of $\mathcal{E}$, and take the limit $V \to 0$ to obtain

$$div \, \mathbf{E} - \frac{\rho}{\epsilon_0} = 0 \text{ , or } div \, \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4}$$

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[1]You will learn how to find solutions to PDEs if you took a course in PDEs.

Equation (4) is Gauss' law in differential form, and is first of Maxwell's four equations.

2. Gauss' Law for magnetic fields in differential form

We learn in Physics, for a magetic field $\mathbf{B}$, the magnetic flux through any closed surface is zero because there is no such thing as a magnetic charge (i.e. monopole). Whenever, there is a north pole, you have a south pole in a magnetic field. So,

$$\int\int_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{5}$$

With the same steps as for the case of electric field, except $\mathbf{B}$ replaces $\mathbf{E}$ and ρ is replaced by 0, it follows from (5) that

$$div\ \mathbf{B} = 0 \tag{6}$$

This is Maxwell's second equation, or Gauss' law for magnetic field in differential form.

3. Faraday's law

We also learn in Physics that if you take a loop and calculate the rate of change of magnetic flux through a near closed loop (imagine we keep the circuit open slightly), it is equal to the induced potential difference V_i, *i.e.*

$$V_i = \frac{d}{dt}\left\{\int\int_S \mathbf{B} \cdot \mathbf{dS}\right\}, \tag{7}$$

where S is the surface enclosed by C and oriented positively with respect to the C. On the otherhand, potential difference is the is the work done to bringing a point test charge from one end of the loop to another, and is therefore given by

$$V_i = -\oint_C \mathbf{E} \cdot \mathbf{dr} \tag{8}$$

Therefore, we obtain Faraday's law in the form

$$\oint_C \mathbf{E} \cdot \mathbf{dr} = -\frac{d}{dt}\left\{\int\int_S \mathbf{B} \cdot \mathbf{dS}\right\}, = -\left\{\int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}\right\}, \tag{8}$$

Using Stokes Theorem, we obtain

$$\oint_C \mathbf{E} \cdot \mathbf{dr} = \int\int_S curl\ \mathbf{E} \cdot \mathbf{dS} \tag{9}$$

Therefore, it follows from (8) that

$$\int\int_S \left\{curl\ \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right\} \cdot \mathbf{dS} = 0$$

Dividing the above answer by A, which is the area of S and taking the limit $A \to 0$ with S chosen to be a planar surface with normal $\mathbf{n}$, we derive[2]

$$\mathbf{n} \cdot \left\{ curl\ \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right\} = 0$$

Since this is true for any orientation of loop C, and therefore for any choice of direction for $\mathbf{n}$, it follows that

$$curl\ \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{10}$$

This is Faraday's law in differential form, or Maxwell's third equation.

4. Ampere's law and correction: Maxwell's fourth equation

You learn Ampere's law in Physics, which states that the loop integral of the magnetic field $\mathbf{B}$ is related to the current I enclosed by the loop C:

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 I \tag{11}$$

Let's assume that the current is distibuted across the cross-section of the loop, with current density $\mathbf{j}$. This density is a vector since it has a direction attached to it (the direction of current flow). So, the total current I enclosed by C is

$$I = \int \int_S \mathbf{j} \cdot \mathbf{dS} \tag{12}$$

Therefore, Ampere's law becomes:

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \int \int_S \mu_0 \mathbf{j} \cdot \mathbf{dS}, \tag{13}$$

where S is a surface whose boundary is C. Using Stokes' Theorem on the left hand side of (13), we obtain

$$\int \int_S \{ curl\ \mathbf{B} - \mu_0 \mathbf{j} \} \cdot \mathbf{dS} = 0$$

Since this is true for arbitrary S, by shrinking C to smaller and smaller loop around a fixed point and dividing by the area of S, we obtain in a manner that should be familiar by now:

$$\mathbf{n} \cdot \{ curl\ \mathbf{B} - \mu_0 \mathbf{j} \} = 0.$$

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[2]This process is similar to that done in class when we derived the physical meaning of $curl\ \mathbf{F}$

Since this is true for any orientation of loop C, and therefore for any direction of $\mathbf{n}$, it follow that

$$curl\ \mathbf{B} = \mu_0\mathbf{j} \tag{14}$$

This is Ampere's law in differential form.

However, equation (14) is not quite correct as Maxwell noted a lack of symmetry between electric and magnetic field in equations (10) and (14) in a region where current density $\mathbf{j} = 0$. He postulated that similar to a changing magnetic field causing an electric field, a changing electric field should also cause a magnetic field. With Maxwell's correction equation (14) is replaced by

$$curl\ \mathbf{B} = \mu_0\left\{\mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right\} \tag{15}$$

This is the fourth of Maxwell's equations.

It is to be noted that $\mu_0\epsilon_0 = \frac{1}{c^2}$, where c is the speed of light. So, the term in (15) that Ampere's law (14) does not include is $\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$, which is small under conditions when Ampere's law was tested. However, this is not the case for electro-magnetic radiation and the correction by Maxwell is extremely important.