1a. Sketch and identify the graph of the following equation in cylindrical coordinates \( w - 1 = 2r \).

**Solution:** Converting into Cartesian coordinates, we have the surface \( z - 1 = 2\sqrt{x^2 + y^2} \). On the \( y-z \) plane, we have for \( x > 0 \), \( z = 1 + 2x \), which is the half-line (since \( x \geq 0 \)) with slope 2 with intercept 1. Since \( z \) depends only on the distance \( r \), the graph is the surface of revolution obtained by rotating this half-line around the \( z \)-axis. We obtain a cone with vertex on the \( z \)-axis at \( z = 1 \).

![Figure 1. Sketch of cone \( z = 1 + 2\sqrt{x^2 + y^2} \)](image)

1b. Determine the angle between the vectors \( 10\mathbf{i} - \mathbf{j} + 14\mathbf{k} \) and \( \mathbf{i} + \mathbf{j} + \mathbf{k} \).

**Solution:** The angle between the two vectors satisfies:

\[
\cos \theta = \frac{(10, -1, 14) \cdot (1, 1, 1)}{\sqrt{10^2 + 1^2 + 14^2} \cdot \sqrt{1^2 + 1^2 + 1^2}} = \frac{23}{\sqrt{891}}
\]

So, \( \theta = \arccos \left( \frac{23}{\sqrt{891}} \right) \approx 0.69 \) radians \( \approx 39.6^\circ \).
2a. Using vectors, show that the diagonals of a parallelogram bisect each other.

Solution: Consider the parallelogram shown. Note vector $\mathbf{OP} = t\mathbf{OC}$ and $\mathbf{AP} = s\mathbf{AB}$ where $t, s$ are some scalar multiples to be determined. We are done if we can show $t = s = 1/2$. Note from the figure, 

$$\mathbf{OC} = \mathbf{a} + \mathbf{b}, \text{ and } \mathbf{AB} = \mathbf{b} - \mathbf{a}, \text{ OP } = \mathbf{OA} + \mathbf{AP}$$

So,

$$\mathbf{OP} = t(\mathbf{a} + \mathbf{b}) = \mathbf{a} + s(\mathbf{b} - \mathbf{a})$$

Therefore, bringing everything on one side $(t + s - 1)\mathbf{a} + (t - s)\mathbf{b} = 0$. Since the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, $(t + s - 1) = 0$ and $t - s = 0$. So, $t = s$. From the first relation $2s - 1 = 0$. Therefore, $s = t = \frac{1}{2}$.

2b. Determine an equation of the plane passing through $(1, 0, 2), (0, 1, 1)$ and $(0, 1, 3)$.

Solution: The plane contains the vectors $\mathbf{a} = (0 - 1, 1 - 0, 1 - 2) = (-1, 1, -1)$ and $\mathbf{b} = (0 - 0, 1 - 1, 3 - 1) = (0, 0, 2)$. Therefore, just computing, a normal vector $\mathbf{N} = \mathbf{a} \times \mathbf{b} = (2, 2, 0)$. So, equation of plane $2(x - 1) + 2y = 0$ or $x + y = 1$. 
3a. Assume that the height of a mountain is given by \( h(x, y) = e^x + y^2 \) in some appropriate length units. Which direction should someone located at \((2, 1, e^3)\) head in order to climb at the steepest rate? What is the maximal rate of climb?

**Solution:** The direction to proceed will be along the vector \( \nabla e^x + y^2 \) evaluated at \((2, 1)\), or along \( \left( e^x + y^2, 2ye^x + y^2 \right) \) when \((x, y) = (2, 1)\), which is the same as along \((1, 2)\). Maximal rate of climb (or the steepest slope) is \( ||\nabla h|| \) at \((1, 2)\), which is \( \sqrt{e^6 + 4e^6} = e^3\sqrt{5} \).

3b. Determine the equation of the tangent plane to \( z^3 + xyz = 33 \) at \((1, 2, 3)\).

**Solution:** Note that the surface given is a level surface of the function \( g(x, y, z) = z^3 + xyz \) and that \( \nabla g = (yz, xz, 3z^2 + xy) \), which at \((1, 2, 3)\) equals \((6, 3, 29)\) and is normal to the tangent plane through \((1, 2, 3)\). Therefore, equation of tangent plane is \( 6(x-1) + 3(y-2) + 29(z-3) = 0 \).
4a. Calculate the arclength of the helix \((\sin(2t), \cos(2t), t + 1)\) between \(t = 0\) and \(t = \pi\).

**Solution:** \(f'(t) = (2 \cos(2t), -2 \sin(2t), 1)\). So, \(\|f'(t)\| = \sqrt{4 \cos^2(2t) + 4 \sin^2(2t) + 1} = \sqrt{5}\). So, arclength of the helix between specified points is

\[
\int_0^\pi \|f'(t)\|\, dt = \sqrt{5} \int_0^\pi dt = \sqrt{5} \pi
\]

4b. Calculate the work done by force \(F = (x, y)\) on a particle moving on a straight line from \((0, 1)\) to \((2, 3)\).

**Solution.** The straight line parametrization \(f(t)\) for the path between two points is \(f(t) = (0, 1) + t(2 - 0, 3 - 1) = (2t, 1 + 2t)\), with \(0 \leq t \leq 1\) (you can double check to see that \(t = 0\) and \(t = 1\) correspond to the two end points given). So, we have work done as

\[
W = \int_0^1 (x, y) \cdot f'(t)\, dt = \int_0^1 ((2t)2 + (1 + 2t)(2))\, dt = \int_0^1 (2 + 8t)\, dt = [2t + 4t^2]_0^1 = 6
\]
5a. Evaluate \( \int \int_R x \, dA \) where \( R \) is the region inside \( x^2 + (y - 1)^2 = 4 \).

**Solution:** Note the region is the interior of a circle of radius 2 centered at \((0, 1)\). If we treat that as an \( x \)-simple region, then we note the left and right curves are given by \( x = -\sqrt{4 - (y - 1)^2} = h_1(y) \) and \( x = \sqrt{4 - (y - 1)^2} = h_2(y) \). Now maximum and minimum values of \( y \) attained when \((y - 1)^2 = 4\), or \( y - 1 = \pm 2\); so \( y \) ranges from \(-1\) to \(3\). Therefore, we have

\[
\int \int_R x \, dA = \int_{-1}^{3} \int_{-\sqrt{4-(y-1)^2}}^{\sqrt{4-(y-1)^2}} x \, dx \, dy = 0,
\]

since \( x \) is an odd function integrated over a symmetric interval \((-\sqrt{4 - (y - 1)^2}, \sqrt{4 - (y - 1)^2})\) about the origin.

5b. Set up an integral with appropriate limits for computing the mass of a solid bounded by \( x = 0, y = 0, z = 0 \) and \( x + y + 2z = 2 \) if the density of the solid is proportional to the distance from the \( x \)-\( y \)-plane.

**Solution:** From the problem statement, density \( \rho = k|z| \), for some constant of proportionality \( k \). Since from the problem statement, the region \( S \) is the the inside of a tetrahedron bounded by the coordinate planes and the plane \( x + y + 2z = 2 \), we may treat this as a \( z \)-simple region with \( 0 \leq z \leq \frac{1}{2}(2 - x - y) \) and the projection of this region in the \( x - y \) plane is the triangle bounded by the lines \( x = 0, y = 0 \) and \( x + y = 2 \). We can treat this region as a \( y \)-simple region in the plane with lower curve \( y = 0 \) and upper-curve \( y = 2 - x \), while minimum and maximum values of \( x \) are between \( 0 \) and \( 2 \). Therefore, noting that \( z \geq 0 \) in \( S \), total mass

\[
M = k \int_0^2 \int_0^{2-x} \int_0^{2-x-y} z \, dz \, dy \, dx.
\]