Solution to Set 4, due Friday April 23

(1) (Section 5.1, Problem 2) Explain why the path parametrized by $\mathbf{f}(t) = (t, |t-1|)$ is not smooth. **Note this is true more specifically if the interval of t contains $t = 1$**
Solution: Define $\mathbf{f}(t) = (f_1, f_2)$. Then, $f_2(t) = |t-1|$ is not differentiable at $t = 1$ hence the curve is not smooth.

(2) (Section 5.1, Problem 4) Show that $\mathbf{f}(t) = (e^{-t}, 1+e^t)$, $0 \le t < \infty$ and $\mathbf{g}(t) = \left(t^3 - 2, \frac{t^3-1}{t^3-2}\right)$ for $2^{1/3} < t \le 3^{1/3}$ parametrizes the same curve.
Solution: Note if $\mathbf{x} = (x, y) = \mathbf{f}(t)$, then $x = e^{-t}$, $y = 1 + e^t$. Eliminating t between them, note that $y = 1 + 1/x$. Further, when $t = 0$, $(x, y) = (1, 2)$. When $t \to +\infty$, we have $(x, y) \to (0, +\infty)$.
On the otherhand, if $\mathbf{x} = (x, y) = \mathbf{g}(t)$, then $x = t^3 - 2$, then $y = \frac{t^3-1}{t^3-2}$. Eliminating t^3 between the two equations, we have $t^3 = x + 2$, so, $y = \frac{x+2-1}{x} = 1 + 1/x$. Further, when $t \to 2^{1/3}$ from above, then $(x, y) \to (0, +\infty)$ and when $t = 3^{1/3}$, we have $(x, y) = (3 - 2, \frac{3-1}{3-2}) = (1, 2)$.
Thus, the curves in each case are the same with the same end points. The only difference is that the curves are traversed in opposite directions as t increases. In the first case, we start from (1, 2) and end up at $(0, +\infty)$; where as in the second case, the starting and ending points are $(0, \infty)$ and (1, 2) respectively.

(3) (Section 5.1, Problem 12) Compute the length of the path $\mathbf{f}(t) = (e^t \cos t, e^t \sin t, e^t)$ for $0 \le t \le 3$.
Solution: Note that $\mathbf{f}'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t)$. So,

$$\|\mathbf{f}'(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}} = e^t\sqrt{2\sin^2 t + 2\cos^2 t + 1} = \sqrt{3}e^t$$

So, arclength equals

$$\int_0^3 \sqrt{3}e^t dt = \sqrt{3}\left[e^t\right]_0^3 = \sqrt{3}\left[e^3 - 1\right]$$

(4) Section 5.1, Problem 15 Find the length of one arch of the cycloid $\mathbf{f}(t) = [a(t - \sin t), a(1 - \cos t)]$.
Solution: Note one arch of a cycloid corresonds to $0 \le t \le 2\pi$. Note $\mathbf{f}'(t) = [a(1 - \cos t), a \sin t]$. So, we have

$$\|\mathbf{f}'(t)\| = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} = a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = a\sqrt{2}\sqrt{1 - \cos t}$$

So, arclength is $\sqrt{2}a\int_0^{2\pi}\sqrt{1-\cos t}dt$. To calculate the integral, recall that $1-\cos 2\theta = 2\sin^2\theta$. Taking $\theta = t/2$,$1-\cos t = 2\sin^2\frac{t}{2}$. So, arclength is

$$2a\int_0^{2\pi}\sin\frac{t}{2}dt = -4a\left[\cos\frac{t}{2}\right]_0^{2\pi} = 8a$$

(5) Section 5.1, Problem 18 Find the arclength function for the spiral parametrized by $\mathbf{f}(t) = (t\cos t, t\sin t)$ for $t \geq 0$. We note that $\mathbf{f}'(t) = (\cos t - t\sin t, \sin t + t\cos t)$. So,

$$\|\mathbf{f}'(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$
$$= \sqrt{\cos^2 t + t^2\sin^2 t - 2t\cos t\sin t + \sin^2 t + t^2\cos^2 t + 2t\cos t\sin t} = \sqrt{1+t^2}$$

So, arclength function (which is defined to be the arclength from $t=0$ to a variable point t) is given by $L(t)$ (or I prefer using $s(t)$)

$$s(t) = \int_0^t \sqrt{1+\tau^2}d\tau$$

To integrate this, the easiest method would be to use hyperbolic sine substitution, $\tau = \sinh\theta$. Note that $\theta = 0$ corresponds to $t = 0$j. Noting that $d\tau = \cosh\theta d\theta$ and $\sqrt{1+\sinh^2\theta} = \cosh\theta$, we have $\sqrt{1+\tau^2}d\tau = \cosh^2\theta d\theta$. We now use the identity $\cosh^2\theta = \frac{1}{2}(1+\cosh 2\theta)$ So,

$$s(t) = \frac{1}{2}\int_0^\theta [1+\cosh(2\theta)]\,d\theta = \frac{\theta}{2}+\frac{1}{4}\sinh(2\theta) = \frac{1}{2}\theta + \frac{1}{2}\sinh\theta\cosh\theta$$
$$= \frac{1}{2}\sinh^{-1}t + \frac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta} = \frac{1}{2}\sinh^{-1}t + \frac{1}{2}t\sqrt{1+t^2}$$

The problem is solved. But, let me show you an alternate method, just in case you are not comfortable with hyperbolic sine and cosine functions and their identities.

Alternately, we could use integration by parts and note

$$I \equiv \int\sqrt{1+t^2}dt = t\sqrt{1+t^2} - \int t d[\sqrt{1+t^2}] = t\sqrt{1+t^2} - \int\frac{t^2}{\sqrt{1+t^2}}dt$$
$$= t\sqrt{1+t^2} - \int\frac{t^2+1}{\sqrt{1+t^2}}dt + \int\frac{dt}{\sqrt{1+t^2}} = t\sqrt{1+t^2} - I + \int\frac{dt}{\sqrt{1+t^2}}$$

So,

$$2I = t\sqrt{1+t^2} + \int\frac{dt}{\sqrt{1+t^2}}$$

Now let's calculate the last integral above. We can substitute $t = \tan\phi$, then $\frac{dt}{\sqrt{1+t^2}} = \sec\phi d\phi$ and recalling that $\int \sec\phi d\phi = \ln(\tan\phi + \sec\phi)$, we have

$$\int \frac{dt}{\sqrt{1+t^2}} = \ln(\tan\phi + \sec\phi) = \ln\left(t + \sqrt{1+t^2}\right)$$

So, from above, we have

$$\int \sqrt{1+t^2}dt \equiv I = \frac{t}{2}\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right) + C$$

Since arc length should zero when $t = 0$, we have $s(0) = 0 = C$. Therefore,

$$s(t) = \frac{t}{2}\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)$$

This result may look different from (**??**), but it is the same since $\sinh^{-1} t = \ln\left(t + \sqrt{1+t^2}\right)$. To show this define $\gamma = \sinh^{-1} t$. Then $t = \sinh\gamma = \frac{1}{2}e^{\gamma} - \frac{1}{2}e^{-\gamma}$. If we define $y = e^{\gamma}$, then we have from above $t = \frac{y}{2} - \frac{1}{2y}$. Multiplying through $2y$, we have $2yt = y^2 - 1$. Solving the quadratic, $y = t + \sqrt{1+t^2}$, we take the positive root, since $y = e^{\gamma} > 0$ and with the negative square-root, we would have $y < 0$. So, noting $y \equiv e^{\gamma}$, we have $\gamma = \ln\left(t + \sqrt{1+t^2}\right)$. But since $\sinh^{-1} t \equiv \gamma$, we have the equality between $\sinh^{-1} t$ and $\ln\left(t + \sqrt{1+t^2}\right)$ as claimed and the two results for the arclength function $s(t)$ is one and the same.

(6) (Section 5.2, Problem 3) Calculate $\int_C (16x - y^2 + 49)dL$, where C is parametrized by $\mathbf{f}(t) = (t^2+1, 4t+7)$ for $0 \le t \le 3$. Note that $\mathbf{f}'(t) = (2t, 4)$ and so, $\|\mathbf{f}'(t)\| = \sqrt{16+4t^2}$. Note that if $\mathbf{x} = (x, y) = \mathbf{f}(t)$, then $x = 1 + t^2$, $y = 4t + 7$, $dL = \|\mathbf{f}'(t)\|dt = 2\sqrt{4+t^2}dt$. so,

$$\int_C (16x - y^2 + 49)dL = \int_0^3 \left(16[1+t^2] - [4t+7]^2 + 49\right) 2\sqrt{4+t^2}dt$$

$$= 2\int_0^3 [16 - 56t]\sqrt{4+t^2}dt = 32\int_0^3 \sqrt{4+t^2}dt - 112\int_0^3 t\sqrt{4+t^2}dt$$

In the second integral above, we substitute $u = 4 + t^2$ and note $2tdt = du$. Note also that $t = 0$ corresponds to $u = 4$ and $t = 3$ corresponds to $u = 4 + 3^2 = 13$. So, we obtain

$$-112\int_0^3 t\sqrt{1+t^2}dt = -56\int_4^{13} \sqrt{u}du = -\frac{(56)(2)}{3}\left[u^{3/2}\right]_4^{13} = -\frac{112}{3}\left[(13)^{3/2} - 8\right]$$

Using methods for the last problem, we have

$$\int \sqrt{4+t^2} = \frac{t}{2}\sqrt{4+t^2} + 2\ln\left(\frac{t}{2} + \sqrt{1+\frac{t^2}{4}}\right)$$

So, using this and the above result, we have

$$\int_C (16x - y^2 + 49)dL = -\frac{112}{3}\left[(13)^{3/2} - 8\right] + 32\left[\frac{t}{2}\sqrt{4+t^2} + 64\ln\left(\frac{t}{2} + \sqrt{1+\frac{t^2}{4}}\right)\right]_0^3$$

$$= -\frac{112}{3}\left[(13)^{3/2} - 8\right] + 48\sqrt{13} + 64\ln\left(\frac{3}{2} + \sqrt{1+\frac{9}{4}}\right) = -\frac{1312}{3}\sqrt{13} + \frac{896}{3} + 64\ln\left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right)$$

(7) (Section 5.2, Problem 5) Calculate $\int_C \frac{x_1+x_2}{x_3-x_4} dL$ where $\mathbf{f}(t)$ is the line segment in 4-D joining $(6,0,3,1)$ to $(5,1,5,3)$.
Solution: Note that in any dimension equation of straight line through a given point $\mathbf{x}_0$ that is parallel to $\mathbf{m}$ is is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{m}$. In this case $\mathbf{x}_0 = (6,0,3,1)$ and $\mathbf{m} = (5,1,5,3) - (6,0,3,1) = (-1,1,2,2)$. So, equation for straight line between two points is $\mathbf{x} = (6,0,3,1) + t(-1,1,2,2) = (6-t, t, 3+2t, 1+2t) = \mathbf{f}(t)$, where $0 \le t \le 1$ covers the line segment between two points. Note $\mathbf{f}'(t) = (-1,1,2,2)$ and $\|\mathbf{f}'(t)\| = \sqrt{1^2+1^2+2^2+2^2} = \sqrt{10}$. So,

$$\int_C \frac{x_1+x_2}{x_3-x_4} dL = \int_0^1 \frac{(6-t)+t}{(3+2t)-(1+2t)}\sqrt{10}dt = 3\sqrt{10}\int_0^1 dt = 3\sqrt{10}$$

(8) (Section 5.2, Problem 7) Find the mass of a wire in the shape of a helix traced by $\left(\cos t, \sin t, \frac{t}{\pi}\right)$ $\pi \le t \le 3\pi$ if its density is proportional to the distance of the point to the $x-y$ plane.
Solution: From the statement the density (mass per unit length in this case) $\rho = k|z|$ for some constant of proportionality k. W need to calculate $M = \int_C k|z|dL$ We note that

$$\mathbf{x} = (x,y,z) = \mathbf{f}(t) = \left(\cos t, \sin t, \frac{t}{\pi}\right)$$

and so $x = \cos t$, $y = \sin t$ and $z = \frac{t}{\pi}$ on the curve C for $\pi \le t \le 3\pi$. Note $z > 0$ so, $|z| = z$. Further, $\|\mathbf{f}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{\pi^2}} = \frac{\sqrt{\pi^2+1}}{\pi}$. So,

$$M = \frac{k}{\pi^2}\sqrt{1+\pi^2}\int_\pi^{3\pi} t dt = \frac{k}{\pi^2}\sqrt{1+\pi^2}\left[t^2/2\right]_\pi^{3\pi} = 4k\sqrt{1+\pi^2}$$

(9) (Section 5.2, Problem 14) Evaluate the line integral of the vector field $\mathbf{F}(x, y) = (y, y + 1 - x^2)$ over the path C that consists of the segments from (5, -1) to (5, 2) and from (5, 2) to (0, 2).
Solution: We note that line integral over the path C (which may be thought of as the total work by a force $\mathbf{F}$ for displacement along C) is the sum $\int_{C_1} \mathbf{F}\cdot\mathbf{dx} + \int_{C_2} \mathbf{F}\cdot\mathbf{dx}$, where C_1 is the straigth line path from (5, -1) to (5,2) and C_2 is the straight line path from (5, 2) to (0, 2). On C_1, $\mathbf{x} = (x, y) = (5, -1) + t(0, 3) = (5, -1 + 3t) = \mathbf{f}(t)$. So, $\mathbf{dx} = \mathbf{f}'(t)dt = (0, 3)dt$ and on C_1,

$$\int_{C_1} \mathbf{F}\cdot\mathbf{dx} = \int_0^1 \left(-1 + 3t, -1 + 3t + 1 - 5^2\right)\cdot(0, 3)dt$$

$$= \int_0^1 3(3t - 25)dt = \left[\frac{9}{2}t^2 - 75t\right]_0^1 = \frac{9}{2} - 75 = -\frac{141}{2}$$

On C_2 between from (5,2) to (0, 2), we have $\mathbf{x} = (x, y) = (5, 2) + t(-5, 0) = (5 - 5t, 2) = \mathbf{g}(t)$ for $0 \le t \le 1$. Now, we have $\mathbf{dx} = \mathbf{g}'(t)dt = (-5, 0)dt$ and on C_2, $\mathbf{F} = (2, 2 + 1 - (5 - 5t)^2)$. So, we have

$$\int_{C_2} \mathbf{F}\cdot\mathbf{dx} = \int_0^1 \left(2, 3 - (5 - 5t)^2\right)\cdot(-5, 0)dt = \int_0^1 (-10)dt = [-10t]_0^1 = -10$$

So, total work done

$$\int_C \mathbf{F}\cdot\mathbf{dx} = \int_{C_1} \mathbf{F}\cdot\mathbf{dx} + \int_{C_2} \mathbf{F}\cdot\mathbf{x} = -\frac{141}{2} - 10 = -\frac{161}{2}$$

(10) (Section 5.2, Problem 19) Evaluate $\int_C (x^2 + yz)dx + zdy + (y - x)dz$ for C given by $\mathbf{x} = (t, 2t - 1, -8t + 2)$ for $0 \le t \le 1$.
Solution: We note that $x = t$, $dx = dt$, $y = 2t - 1$, $dy = 2dt$, $z = -8t + 2$, $dz = -8dt$. So,

$$\int_C (x^2 + yz)dx + zdy + (y - x)dz$$

$$= \int_0^1 \left[(t^2 + (2t - 1)(-8t + 2))dt + (-8t + 2)(2dt) + (2t - 1 - t)(-8dt)\right]$$

$$= \int_0^1 \left[-15t^2 - 12t + 10\right] dt = \left[-5t^3 - 6t^2 + 10t\right]_0^1 = -1$$

(11) (Section 5.2, Problem 22) Find an expression for the work done by the gravitational field $\mathbf{F} = (0, 0, -g)$ near the earth's surface on a particle of mass m that moves from the orgin to position $\mathbf{x}_0 = (a, b, c)$ along
a. the line segment from the origin to this point.

b. the curve consisting from (0, 0, 0) to $(a, 0, 0)$, $(a, 0, 0)$ to $(a, b, 0)$, and $(a, b, 0)$ to (a, b, c).
Solution to a. Note on the line segment $\mathbf{x} = t(a, b, c) = (at, bt, ct)$, for $0 \leq t \leq 1$. So, work done

$$\int_0^1 (0, 0, -mg) \cdot (adt, bdt, cdt) = -mgc \int_0^1 dt = -mgc$$

Solution to b. On the line segment from (0, 0, 0) to $(a, 0, 0)$ where $\mathbf{x} = (at, 0, 0)$, we have no work since Force $(0, 0, -mg)$ is orthogonal to $\mathbf{dx} = (adt, 0, 0)$. There is also no work done on the line segment from $(a, 0, 0)$ to $(a, b, 0)$ since on that line segement $\mathbf{x} = (a, 0, 0) + t(0, b, 0)$ and $d\mathbf{x} = (0, b, 0)dt$, which is again perpendicular to force $(0, 0, -mg)$. So, the only work done is on the line segment between $(a, b, 0)$ and (a, b, c). In this case $\mathbf{x} = (a, b, 0) + t(0, 0, c)$. So $\mathbf{dx} = (0, 0, c)dt$. So, work done

$$\int_0^1 (0, 0, -mg) \cdot (0, 0, c)dt = -mgc \int_0^1 dt = -mgc,$$

same work in case (**a.**).

(12) (Section 5.2, Problem 24) Show that $\int_{-C} \mathbf{F} \cdot \mathbf{dx} = -\int_C \mathbf{F} \cdot \mathbf{dx}$.
Solution: Take a representation for curve C: $\mathbf{x} = \mathbf{f}(t)$ for $a \leq t \leq b$. So,

$$\int_C \mathbf{F} \cdot \mathbf{dx} = \int_a^b \mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{f}'(t)dt$$

Since curve $-C$ is defined to be same path as C, but traversed in the opposite direction, *i.e.* start at point corresponding to $t = b$ and end at point corresponding to $t = a$. So,

$$\int_{-C} \mathbf{F} \cdot \mathbf{dx} = \int_b^a \mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{f}'(t)dt = -\int_a^b \mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{f}'(t)dt = -\int_C \mathbf{F} \cdot \mathbf{dx}$$

(13) (Section 5.2, Problem 25) In example 5.2.7, suppose that the rocket is propelled straight up from the earth's surface to an altitude R_2 which is much larger than R_e (radius of the earth), *ie.* $R_2 \to +\infty$. Approximately how much energy will be required.
Solution: Note work done by the force that propels the object to position R_2 from the surface of the earth is (worked in class, with slight change in notation $R_1 + R_e -> R_1$ and $R_2 + R_e -> R_2$)

$$\frac{GMm}{R_e + R_1} - \frac{GMm}{R_e + R_2} \to \frac{GMm}{R_e + R_1} \text{ as } R_2 \to +\infty$$

The limit is the amount of energy expended.

(14) (Section 5.3, Problem 15) Calculate the iterated integral $\int_0^1 \int_{e^{-x}}^{e^x} \frac{\ln y}{y} dydx$.
Solution: Note that to calculate $\int \frac{\ln y}{y} dy$ we substitute $u = \ln y$, then $du = \frac{dy}{y}$. So $\int \frac{\ln y}{y} dy = \int \frac{du}{u} = \ln|u| = \ln|\ln y|$. Therefore

$$\int_0^1 \int_{e^{-x}}^{e^x} \frac{\ln y}{y} dydx = \int_0^1 [\ln|\ln y|]_{y=e^{-x}}^{y=e^x} dx$$
$$= \int_0^1 \left[\ln|\ln e^x| - \ln|\ln(e^{-x}|)\right] dx = \int_0^1 2(x-x)dx = 0$$

(15) (Section 5.3, Problem 16) Calculate the iterated integral $\int_0^{\frac{\pi}{8}} \int_0^y \sec^2(x+y)\, dxdy$.
Solution: We note that $\int \sec^2(x+y)dx = \tan(x+y)$. So,

$$\int_0^{\frac{\pi}{8}} \int_0^y \sec^2(x+y)\, dxdy = \int_0^{\frac{\pi}{8}} [\tan(x+y)]_{x=0}^{x=y} dy = \int_0^{\frac{\pi}{8}} [\tan(2y) - \tan y]\, dx$$
$$= \left\{\frac{1}{2}\ln\sec[2y] - \ln\sec[y]\right\}_0^{\pi/8} = \frac{1}{2}\ln\left(\sec\frac{\pi}{4}\right) - \ln\left(\sec\frac{\pi}{8}\right)$$

(16) (Section 5.3, Problem 19) Calculate the double integral (area integral) $\int\int_R ye^x dA$ where R is the region bounded by the parabola $x = y^2$ and the line $x = 5y$.

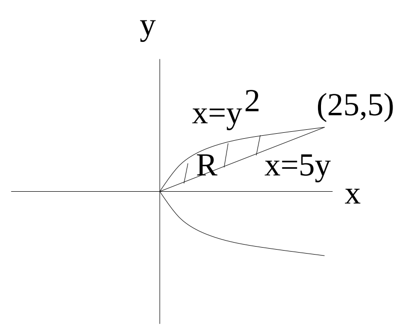

FIGURE 1. Region R bounded by $x = 5y$ and $x = y^2$

Solution: Note the region between the two curves $x = y^2$ and $x = 5y$ intersect when $y^2 = 5y$, implying $y = 0$ (in which case $x = 0$ or $y = 5$ in which case $x = 25$. So, the intersection of two curves have coordinates $(0,0)$ and (25, 5) as shown in the figure. Note R can be treated as an x-simple region, with left curve $x = y^2$ and right curve $x = 5y$. So, we have

$$\int\int_R ye^x dA = \int_0^5 \int_{y^2}^{5y} ye^x dxdy = \int_0^5 [ye^x]_{x=y^2}^{x=5y} dy = \int_0^5 ye^{5y}dy - \int_0^5 ye^{y^2} dy$$

We note that on integration by parts

$$\int ye^{5y}dy = \frac{1}{5}\int yd[e^{5y}] = \frac{y}{5}e^{5y} - \frac{1}{5}\int e^{5y}dy = \frac{y}{5}e^{5y} - \frac{1}{25}e^{5y}$$

Also, note on substituting $u = y^2$ (in which case $2ydy = du$), we have

$$\int ye^{y^2} dy = \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{y^2}$$

So, from above,

$$\int\int_R ye^x dA = \left\{-\frac{e^{y^2}}{2} + \frac{y}{5}e^{5y} - \frac{e^{5y}}{25}\right\}_{y=0}^{y=5} = -\frac{e^{25}}{2} + e^{25} - \frac{e^{25}}{25} + \frac{1}{2} + \frac{1}{25} = \frac{23}{50}e^{25} + \frac{27}{50}$$

(17) (Section 5.3, Problem 29) Find the volume of the solid bounded above by $x^2 + y^2 + z^2 = 16$ and below by $z = \frac{1}{6}(x^2 + y^2)$(Set up the integral only).

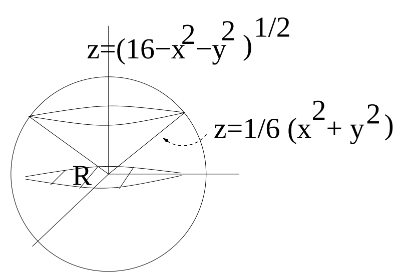

FIGURE 2. Solid bounded above by $x^2 + y^2 + z^2 = 16$ and below by $z = \frac{1}{6}(x^2+y^2)$. Note the projection of solid in the $x - y$ plane gives region R

Solution: The projection region R on the $x-y$ plane is clearly the shadow of the curve where the surface $x^2 + y^2 + z^2 = 16$ intersects to $z = \frac{1}{6}(x^2+y^2)$. Where the intersect, note $x^2+y^2 = 6z = 16 - z^2$. So, $z^2 + 6z - 16 = 0$. So, the positive root of the quadratic $z = 2$. Plugging in this value of z into $x^2 + y^2 = 6z$, we obtain the boundary of region R in the $x - y$ plane to be $x^2+y^2 = 12$, a circle of radius $\sqrt{12}$ centered at $(0, 0)$ in the $x-y$ plane. Treating this R as a y-simple region, we have upper curve $y = \sqrt{12 - x^2} = g_2(x)$ and lower curve $y = -\sqrt{12 - x^2} = g_1(x)$.

Note the range of x is x between $-\sqrt{12}$ and $\sqrt{12}$. At each point $(x, y) in R$, we note that the height function of the column between the two surfaces is $h(x, y) = \sqrt{16 - x^2 - y^2} - \frac{1}{6}(x^2 + y^2)$. So, we have volume

$$V = \int\int_R h(x, y) dA = \int_{-\sqrt{12}}^{\sqrt{12}} \int_{-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \left[\sqrt{16 - x^2 - y^2} - \frac{1}{6}(x^2 + y^2)\right] dydx$$

(18) (Section 5.3, Problem 35) R is the region in the $x - y$ plane bounded by curves $y = x^2 + 1$ and $y = x + 3$ with density at a point proportional to the distance from of that point to the x-axis. Calculate the mass.

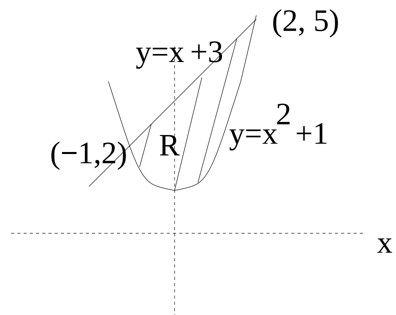

FIGURE 3. Region R bounded by $y = x^2 + 1$ and $y = x + 3$

Solution: From the problem statement density $\sigma(x, y) = k|y|$. Note that the intersection of the two curves occur where $x+3 = x^2 + 1$, or $x^2 - x - 2 = 0$, implying $x = 2$ (corresponding $y = x + 3 = 5$) or $x = -1$ for which $y = x + 3 = 2$. So, the coordinates of the intersection points of two curves are (2,5) and (-1, 2) as shown in the figure. We treat this conveniently as a y-simple curve with a lower curve $y = x^2 + 1 = g_1(x)$ and an upper curve $y = x + 3 = g_2(x)$. The range of x is found by looking at the x-coordinate of the intersection points–so x is between -1 and 2. Therefore, since $y > 0$ in R, mass equals

$$\int\int_R \sigma(x, y) dA = \int_{-1}^{2} \int_{1+x^2}^{x+3} ky dydx = k \int_{-1}^{2} \left[\frac{y^2}{2}\right]_{y=1+x^2}^{y=x+3} dx = \frac{k}{2} \int_{-1}^{2} \left[(x + 3)^2 - (1 + x^2)^2\right] dx$$

$$= \frac{k}{2} \int_{-1}^{2} \left[8 + 6x - x^2 - x^4\right] dx = \frac{k}{2} \left[8x + 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \frac{k}{2} \left[16 + 12 - \frac{8}{3} - \frac{32}{5} + 8 - 3 - \frac{1}{3} - \frac{1}{5}\right] = \frac{117}{10} k$$

(19) (Section 5.3, Problem 39) Reverse the order of integration

$$\int_{\pi/2}^{\pi}\int_{0}^{\sin x} f(x,y)dydx$$

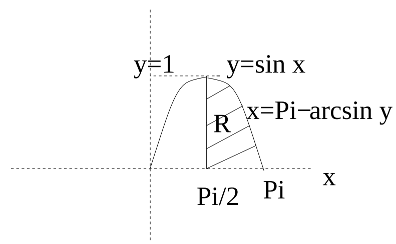

FIGURE 4. Region R bounded between lower curve $y = 0$ and $y = \sin x$, for $\frac{\pi}{2} \leq x \leq \pi$, which is the same as the region between left curve $x = \frac{\pi}{2}$ and right curve $x = \pi - \arcsin y$, with $0 \leq y \leq 1$

Solution: Looking at the limits, we have in the above integral representation of R as a y-simple region with lower curve $y = 0$ and upper curve $y = \sin x$ and we are ranging from $x = \frac{\pi}{2}$ and $x = \pi$. So, we first plot the region R as shown in the figure. If we now treat R as a x-simple region, we have left curve $x = \pi/2 = h_1(y)$ and right curve $x = \pi - \arcsin y = h_2(y)$ (**Note:** it is not $\arcsin y$ which will give you a value for $x \leq \frac{\pi}{2}$) since $x \geq \frac{\pi}{2}$. Since $y = \sin x$ has maximum value 1, and it is attained at $x = \frac{\pi}{2}$, so from the figure, we have range of y between $y = 0$ and $y = 1$. Therefore,

$$\int_{\pi/2}^{\pi}\int_{0}^{\sin x} f(x,y)dydx = \int\int_{R} f(x,y)dA = \int_{0}^{1}\int_{\pi/2}^{\pi-\arcsin y} f(x,y)dxdy$$