Solution to Set 6, Friday May 7th

(1) (Section 6.2, Problem 5) Use Green’s Theorem to evaluate the line integral \( \oint_C (xy^2 + x)dx + (3x + y^2)dy \), where \( C \) is the boundary in the counter-clockwise sense of the region bounded by \( y = 0, y = 1, y = -x \) and \( x = y^2 \).

![Figure 1. Counter-clockwise closed contour C enclosing region R between y = 0, y = 1, y = -x and x = y^2]

**Solution:** Note \( \frac{\partial F_2}{\partial x} = 3 \) and \( \frac{\partial F_1}{\partial y} = 2xy \); so noting that \( F = (F_1, F_2) \) is smooth on inspection, by using Green’s theorem,

\[
\oint_C (xy^2 + x)dx + (3x + y^2)dy = \iint_R (3 - 2xy) dA = \int_0^1 \int_{-y}^y (3 - 2xy) dxdy
\]

\[
= \int_0^1 \left[ 3x - x^2y \right]_{x=-y}^{x=y^2} dy = \int_0^1 \left[ 3y^2 - y^5 + 3y + y^3 \right] dy
\]

\[
= \left[ y^3 - \frac{y^6}{6} + \frac{3y^2}{2} + \frac{y^4}{4} \right]_0^1 = 1 - \frac{1}{6} + \frac{3}{2} + \frac{1}{4} = \frac{31}{12}
\]

(2) (Section 6.2, Problem 8) Can Green’s function be applied to evaluate \( \oint_C \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy \), where \( C \) is a circle of radius 1 centered at the origin. Explain.

**Solution:** Note that \( F = \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) \) and/or its first derivatives are singular (meaning not continuous) when \( x^2 + y^2 = 0 \), i.e. at \( (x, y) = (0, 0) \). Since the contour \( C \) contains the origin \( (0, 0) \), we cannot apply the Green’s Theorem.

**Comment:** If I had asked you to do this line integral, you should do this using parametric representation \( (x, y) = (\cos t, \sin t) \).
with \((dx, dy) = (−\sin t dt, \cos t dt)\) and converting the integration as an integration over \(t\) from 0 to \(2\pi\).

(3) (Section 6.2, Problem 9) Can Green’s Theorem be applied to evaluate
\[
\oint_C \frac{2x}{(x^2 + (y - 2)^2)^2} dx - \frac{2(y - 2)}{(x^2 + (y - 2)^2)^2} dy
\]
where \(C\) is the unit circle.

**Solution:** Note in this case that
\[
\mathbf{F} = \left(\frac{2x}{(x^2 + (y - 2)^2)^2}, \frac{2y}{(x^2 + (y - 2)^2)^2}\right)
\]
and \(\mathbf{F}\) and its derivatives exist and are continuous everywhere except where the denominator term \(x^2 + (y - 2)^2 = 0\), i.e. \((x, y) = (0, 2)\). However, a unit circle centered at the origin \((0, 0)\) does not enclose this singular (“bad”) point. Hence Green’s theorem can indeed be applied.

(4) (Section 6.2, Problem 12) Use the corollary to Green’s theorem to find the area of the indicated region: \(R\) is the cardioid bounded by \(x = \cos \theta - \cos^2 \theta, y = \sin \theta - \cos \theta \sin \theta, 0 \leq \theta \leq 2\pi\) as shown in text Figure 6.2.6.

**Solution:** We have Area
\[
A = -\oint_C ydx = \\
-\int_0^{2\pi} (\sin \theta - \cos \theta \sin \theta) (-\sin \theta + 2 \cos \theta \sin \theta) d\theta \\
\int_0^{2\pi} \left(\sin^2 \theta + \frac{1}{2} \sin^2 (2\theta) - 3 \cos \theta \sin^2 \theta\right) d\theta \\
= \int_0^{2\pi} \left(\frac{1}{2} [1 - \cos (2\theta)] + \frac{1}{4} - \frac{1}{4} \cos 4\theta - 3 \cos \theta \sin^2 \theta\right) d\theta \\
= \left[\frac{3}{4} \theta - \frac{1}{4} \sin (2\theta) - \frac{1}{16} \sin (4\theta) - \sin^3 \theta\right]_0^{2\pi} = \frac{3}{2}\pi
\]

(5) (Section 6.3, Problem 2) Calculate \(\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} d\sigma\), where \(\mathbf{F} = (x \sin^2 y, \cos z, z \cos^2 y)\) and \(S = [0, \pi] \times [0, \pi/2] \times [-\pi, 0]\).

**Solution:** Note \(\mathbf{F}\) is smooth in \(S\) and \(\nabla \cdot \mathbf{F} = \sin^2 y + \cos^2 y = 1\). Therefore, using divergence theorem
\[
\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_S \cos^2 y dV = \int_0^\pi \int_{-\pi}^\pi \left\{\int_0^{\pi/2} 1\right\} dydzdx
\]
Noting that \( \{.\} \) term is a constant, we have answer equal to

\[
\pi^2 \left\{ \int_{0}^{\pi/2} 1 \, dy \right\} = \frac{\pi^3}{2}
\]

(0.1)

(6) (Section 6.3, Problem 3) Calculate \( \int \oint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, d\sigma \), where \( \mathbf{F} = (y - 3xz^2, x^2 + z^2, xy + z^3) \) and \( S \) is the solid bounded above by the sphere \( x^2 + y^2 + z^2 = 1 \) and below by the \( x - y \) plane.

**Solution:** Note that \( \mathbf{F} \) is smooth everywhere since there are no points where \( \mathbf{F} \) or any of its component derivatives are ill-behaved. Further, note that

\[
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (y - 3xz^2) + \frac{\partial}{\partial y} (x^2 + z^2) + \frac{\partial}{\partial z} (xy + z^3) = -3z^2 + 0 + 3z^2 = 0
\]

Therefore, from divergence theorem

\[
\int \oint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int \int \int_S 0 \, dV = 0
\]

(7) (Section 6.3, Problem 4) Calculate \( \int \oint_{\partial S} \mathbf{x} \cdot \mathbf{n} \, d\sigma \), where \( S \) is the tetrahedral solid in the first octant bounded by coordinate planes and \( 6x + 3y + 2z = 6 \).

**Solution:** Note that \( \nabla \cdot \mathbf{x} = \nabla \cdot (x, y, z) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 3 \). So, since \( (x, y, z) \) is clearly smooth, applying divergence theorem,

\[
\int \oint_{\partial S} \mathbf{x} \cdot \mathbf{n} \, d\sigma = \int \int \int_S 3 \, dV = 3 \int \int \int_S dV
\]

Note that the projection of \( S \) on the \( x - y \) plane \( R \) is the triangle formed by \( x = 0, y = 0 \) and \( 6x + 3y = 6 \), i.e. \( y = -2x + 2 \). as \( x \) ranges from 0 to 1. For fixed \( (x, y) \) in \( R \), \( z \) ranges from 0 to \( z = 3 - \frac{3}{2}y - 3x \). So, the answer for the total outwards flux

\[
3 \int \int \int_S dV = 3 \int \int_R \left\{ \int_{y=0}^{3-\frac{3}{2}y-3x} \, dz \right\} \, dA
\]

\[
= 3 \int_0^1 \int_{y=0}^{3-\frac{3}{2}y-3x} \left\{ 3 - \frac{3}{2}y - 3x \right\} \, dy \, dx
\]

\[
= 3 \int_0^1 \left[ 3y - \frac{3}{4}y^2 - 3xy \right]_{y=0}^{y=2-2x} \, dx = 3 \int_0^1 \left[ 3(2 - 2x) - \frac{3}{4}(2 - 2x)^2 - 3x(2 - 2x) \right] \, dx
\]

\[
= 3 \left[ 6x - 3x^2 + (1 - x)^3 - 3x^2 + 2x^3 \right]_{x=0}^1 = 3
\]

(8) (Section 6.3, Problem 5) Calculate \( \int \oint_{\partial S} \frac{\mathbf{x}}{||\mathbf{x}||} \cdot \mathbf{n} \, d\sigma \) where \( S \) is any solid region not containing the origin.
Solution: Note

\[ \nabla \cdot \left( \frac{x}{\|x\|^3} \right) = \nabla \cdot \left( \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \]

\[ = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \]

Note that

\[ \frac{\partial}{\partial x} \left[ \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right] = \frac{1}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3x}{2(x^2 + y^2 + z^2)^{5/2}} \]

\[ = \frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} \]

From changing variables \( x \rightarrow y, y \rightarrow x \), it follows that

\[ \frac{\partial}{\partial y} \left[ \frac{y}{(y^2 + x^2 + z^2)^{3/2}} \right] = \frac{x^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^{5/2}} \]

and again from changing variables \( y \rightarrow z, z \rightarrow y \), it follows

\[ \frac{\partial}{\partial z} \left[ \frac{z}{(z^2 + x^2 + y^2)^{3/2}} \right] = \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}} \]

Combining, we have

\[ \nabla \cdot \left( \frac{x}{\|x\|^3} \right) = \frac{y^2 + z^2 - 2x^2 + x^2 + z^2 - 2y^2 + x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}} = 0 \]

for \((x, y, z) \neq (0, 0, 0)\). Thus, if \( S \) does not contain the origin we can use divergence theorem to prove that

\[ \int \int \int_{S} \nabla \cdot \left( \frac{x}{\|x\|^3} \right) \, dV = 0 \]

(9) (Section 6.3, Problem 15) Let \( \mathbf{F} \) be such that \( \nabla \times \mathbf{F} \) (or \( \text{curl} \mathbf{F} \)) is continuously differentiable on an open connected set \( U \subset \mathbb{R}^3 \). Show that

\[ \int \int_{\partial S} \text{curl} \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0 \]

Solution: We will apply divergence theorem since the conditions are satisfied on \( S \) to obtain.

\[ \int \int_{\partial S} \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = - \int \int_{S} \nabla \cdot (\nabla \times \mathbf{F}) \, dV \]

We will now prove our result by showing that for any smooth \( \mathbf{F} \), \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \). But this was proved in a prior homework problem.
(10) (Section 6.3, Problem 16) Let $U \subset \mathbb{R}^3$ be an open connected set and $f : U \rightarrow \mathbb{R}$ have continuous second partials on $U$. Show that for any $S \subset U$,
\[
\int \oint_{\partial S} (\nabla f) \cdot n \, d\sigma = \int \int \int_S [\nabla^2 f] \, dV
\]

**Solution:** We define $F = \nabla f = (f_x, f_y, f_z)$. From applying divergence theorem
\[
\int \oint_{\partial S} (\nabla f) \cdot n \, d\sigma = \int \int \int_S \nabla \cdot F \, dV = \int \int \int_S [f_{xx} + f_{yy} + f_{zz}] \, dV = \int \int \int_S [\nabla^2 f] \, dV
\]

(11) (Section 6.3, Problem 10) Calculate $\int \oint_{\partial S} y^2 \, dy \wedge dz + z \ln(x^2 + y^2 + 1) \, dx \wedge dy$, where $S$ is the cylindrical solid bounded by $z = 1$, $z = 4$ and $x^2 + y^2 = \frac{1}{4}$.

**Solution** We have from explanation given in class,
\[
\int \oint_{\partial S} y^2 \, dy \wedge dz + z \ln(x^2 + y^2 + 1) \, dx \wedge dy = \int \oint_{\partial S} F \cdot n \, d\sigma
\]
\[
= \int \int \int_S \nabla \cdot F \, dV,
\]
if indeed $F$ is smooth. We have
\[
F = (y^2, 0, z \ln[x^2 + y^2 + 1])
\]
\[
\nabla \cdot F = \frac{\partial y^2}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial}{\partial z} [z \ln(x^2 + y^2 + 1)] = \ln(x^2 + y^2 + 1)
\]
Note $F$ is smooth everywhere since $(x^2 + y^2 + 1) > 0$. So, divergence theorem is indeed valid and our answer is
\[
\int \int \int_S \ln(x^2 + y^2 + 1) \, dV = \int \int R \int_{1/4}^1 \ln(x^2 + y^2 + 1) \, dz \, dA = 3 \int \int R \ln(x^2 + y^2 + 1) \, dA
\]
where $R$ is the projected region of $S$ in the $x - y$ plane, which is the inside of the circle $x^2 + y^2 < \frac{1}{4}$, i.e. $r < \frac{1}{2}$. Using polar coordinates to describe $R$, and substitution of $u = 1 + r^2$, we have
\[
3 \int \int_R \ln(x^2 + y^2 + 1) = 3 \int_0^{2\pi} \int_0^{1/2} \ln(r^2 + 1) \, r \, dr \, d\theta = 6\pi \int_0^{1/2} \ln(r^2 + 1) \, r \, dr
\]
\[
= 3\pi \int_1^{5/4} \ln u \, du = 3\pi [u \ln u - u]_{u=1}^{u=2} = 3\pi \left[ \frac{5}{4} \ln \frac{5}{4} - \frac{5}{4} + 1 \right]
\]
(12) (Section 6.3, Problem 13) Calculate \[ \int \int_{\partial S} 2xy dy \wedge dz - y^2 dz \wedge dx + zdz \wedge dy; \] where \( S \) is the solid portion of the ball \( x^2 + y^2 + z^2 \leq 4 \) that remains after the portion of the inside the cylindrical solid \( x^2 + y^2 \leq 1 \) is removed.

**Solution:** Note that the projection \( R \) of the solid \( S \) as described above in the \( x-y \) plane is described by \( 1 \leq x^2 + y^2 \leq 4 \), or \( 1 \leq r \leq 2 \) in polar coordinates. For each such \((x, y)\) in \( R \), \( z \) ranges from \(-\sqrt{4 - x^2 - y^2}\) to \(\sqrt{4 - x^2 - y^2}\) in the solid region \( S \). Further, using divergence theorem,

\[
\int \int_{\partial S} 2xy dy \wedge dz - y^2 dz \wedge dx + zdz \wedge dy = \int \int_{\partial S} (2xy, -y^2, z) \cdot \mathbf{n} d\sigma
\]

\[
= \int \int_{R} \left\{ \int_{-\sqrt{4 - x^2 - y^2}}^{\sqrt{4 - x^2 - y^2}} [2y - 2y + 1] \, dz \right\} dA = 2\pi \int_{0}^{2\pi} \int_{1}^{2} 2\sqrt{4 - r^2} r dr d\theta
\]

\[
= 2\pi \left[ -\frac{2}{3} (4 - r^2)^{3/2} \right]_{r=1}^{r=2} = \frac{4\pi}{3} \left( \frac{3}{2} \right)^{3/2} = 4\sqrt{3}\pi
\]

(13) (Section 6.3, Problem 19) Let \( S \) be a solid region in \( \mathbb{R}^3 \) and let \( V \) denote its volume. Use the divergence theorem to show that

\[
V = \int \int_{\partial S} xdy \wedge dz = \int \int_{\partial S} ydz \wedge dx = \int \int_{\partial S}zd\sigma
\]

\[
= \frac{1}{3} \left\{ \int \int_{\partial S} xdy \wedge dz + ydz \wedge dx + zdz \wedge dy \right\}
\]

**Solution:** Note from divergence theorem that for any smooth \( \mathbf{F} = (F_1, F_2, F_3) \) which gives \( \nabla \cdot \mathbf{F} = 1 \), we obtain

\[
\int \int_{\partial S} F_1 dy \wedge dz + F_1 dz \wedge dx + F_3 dx \wedge dy = \int \int_{\partial S} \mathbf{F} \cdot \mathbf{n} d\sigma
\]

\[
= \int \int \int_{S} \nabla \cdot \mathbf{F} dV = \int \int \int 1 dV = V
\]

We may choose \( \mathbf{F} = (x, 0, 0) \) for which \( \nabla \cdot \mathbf{F} = 1 \), which proves the first volume formula; \( \mathbf{F} = (0, y, 0) \) (whose divergence is again 1) gives the second volume formula; \( \mathbf{F} = (0, 0, z) \), whose divergence is 1, gives the third, while \( \mathbf{F} = (\frac{x}{3}, \frac{y}{3}, \frac{z}{3}) \) whose divergence is again 1 gives the last volume formula.
(14) Calculate the outwards flux of the vector field

\[ \mathbf{F} = \frac{\mathbf{x}}{||\mathbf{x}||^3} + \frac{\mathbf{x} - (1,0,0)}{||\mathbf{x} - (1,0,0)||^3} \]

through the surface of a sphere of radius 2 centered at the origin.

**Hint:** You may note from definition of fluxes that the total flux due to a field \( \mathbf{F} = \mathbf{G} + \mathbf{H} \) is the sum of flux due to \( \mathbf{G} \) and flux due to \( \mathbf{H} \). Also, you may use any result I derived in class.

**Solution:** Define \( \mathbf{G}(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||^3} \), \( \mathbf{H} = \frac{\mathbf{x} - (1,0,0)}{||\mathbf{x} - (1,0,0)||^3} \). We showed in class that the outwards flux of \( \mathbf{G} \) through any closed surface enclosing the origin is \( 4\pi \). So, we are left with calculating outwards flux of \( \mathbf{H} \) through the surface of the sphere of radius 2 centered at the origin \((0,0,0)\).

First, we note that if \( \mathbf{x} = (x,y,z) \), then

\[ \begin{bmatrix} \frac{x-1}{||(x-1)^2 + y^2 + z^2||^{3/2}} & \frac{y}{||(x-1)^2 + y^2 + z^2||^{3/2}} & \frac{z}{||(x-1)^2 + y^2 + z^2||^{3/2}} \end{bmatrix} \]

Then, calling \( J = (x-1)^2 + y^2 + z^2 \),

\[ \nabla \cdot \mathbf{H} = \frac{\partial}{\partial x} \left[ (x-1)[(x-1)^2 + y^2 + z^2]^{-3/2} \right] + \frac{\partial}{\partial y} \left[ y[(x-1)^2 + y^2 + z^2]^{-3/2} \right] + \frac{\partial}{\partial z} \left[ z[(x-1)^2 + y^2 + z^2]^{-3/2} \right] \]

\[ = J^{-3/2} + (x-1)(-3/2)J^{-5/2}(2)(x-1) + J^{-3/2} + y(-3/2)J^{-5/2}(2)(y) + J^{-3/2} + z(-3/2)J^{-5/2}(2)(z) \]

\[ = 3J^{-3/2} - 3[(x-1)^2 + y^2 + z^2]J^{-5/2} = 3J^{-3/2} - 3J^{-3/2} = 0, \]

except where \( J = 0 \), i.e. \( (x,y,z) = (1,0,0) \). However, though sphere of radius 2 around the origin contains this bad point \((1,0,0)\), we know from the argument in class that the outward flux of \( \mathbf{H} \) through this surface is the same as the outward flux of \( \mathbf{H} \) through a sphere of any radius centered at \((1,0,0)\) since \( \nabla \cdot \mathbf{H} = 0 \) for \( \mathbf{x} \neq (1,0,0) \). However, for a sphere centered at \((1,0,0)\), \( \mathbf{n} = \frac{\mathbf{x} - (1,0,0)}{||\mathbf{x} - (1,0,0)||} \) and \( \mathbf{H} \cdot \mathbf{n} = \frac{(\mathbf{x} - (1,0,0)) \cdot (\mathbf{x} - (1,0,0))}{||\mathbf{x} - (1,0,0)||} = 1 \).

So, since the surface area of the sphere of radius 1 is \( 4\pi \). So,

\[ \int_{\partial S} \mathbf{F} \cdot \mathbf{n} d\sigma = \int_{\partial S} \mathbf{G} \cdot \mathbf{n} d\sigma + \int_{\partial S} \mathbf{H} \cdot \mathbf{n} d\sigma = 4\pi + 4\pi = 8\pi \]