Solution to Set 7, Due May 21st

(1) (Section 6.4, Problem 2) Calculate \( \oint_C 5yz \, dx + (x^2 - y) \, dy + yz \, dz \) using Stokes Theorem, where \( C \) is parametrized by \( \mathbf{x} = (\cos t, \sin t, \cos t - \sin t) \), \( 0 \leq t \leq 2\pi \).

**Solution:** Note \( C \) is the boundary of plane \( z = x - y \equiv h(x, y) \) that is restricted to \( x^2 + y^2 \leq 1 \) (call this \( M \)) since on the boundary \( x^2 + y^2 = 1 \), \( z = x - y \) corresponds to the given parametric representation. Note \( h_x = 1 \), \( h_y = -1 \). Note the projection of \( M \) in the \( x - y \) plane is the interior of the circle \( x^2 + y^2 \leq 1 \), call it \( R \). Since \( \mathbf{F} = (5yz, x^2 - y, yz) \) and so

\[
\nabla \times \mathbf{F} = (\partial_y[yz] - \partial_z(x^2 - y), \partial_z[5yz] - \partial_x[yz], \partial_x[x^2 - y] - \partial_y[5yz])
= (z, 5y, 2x - 5z)
\]

On \( z = x - y \), \( \nabla \times \mathbf{F} = (x - y, 5y, 2x - 5x + 5y) = (x - y, 5y, -3x + 5y) \). Therefore, using Stokes Theorem

\[
\oint_C \mathbf{F} \cdot d\mathbf{x} = \int \int_M (\nabla \times \mathbf{F}) \cdot d\mathbf{n} d\sigma = \int \int_R (x - y, 5y, -3x + 5y) \cdot (-1, 1, 1) dA
= \int \int_R [-x + y + 5y - 3x + 5y] dA = \int \int_R [11y - 4x] dA
= \int_0^1 \int_0^{2\pi} [11r \sin \theta - 4r \cos \theta] d\theta rdr = \int_0^1 [-11r \cos \theta - 4r \sin \theta]_{\theta=0}^{\theta=2\pi} r dr = 0
\]
(2) (Section 6.4, Problem 4). Using Stokes Theorem, calculate
\[ \oint_C x^2dx + y^2dy + z^2dz \]
where \( C \) is parametrized by
\[ \mathbf{x} = (6t, 0, 5t), 0 \leq t \leq 1; \quad \mathbf{x} = (12 - 6t, 0, 3t + 2), 1 \leq t \leq 2; \]
\[ \mathbf{x} = (0, 0, 24 - 8t), 2 \leq t \leq 3 \]

\[ \nabla \times \mathbf{F} = \left( \frac{\partial y}{\partial z}(z^2) - \frac{\partial z}{\partial x}(x^2), \frac{\partial z}{\partial x}(y^2) - \frac{\partial x}{\partial y}(x^2), \frac{\partial x}{\partial y}(y^2) - \frac{\partial y}{\partial z}(x^2) \right) = (0, 0, 0) \]

Therefore, using Stokes Theorem
\[ \oint_C x^2dx + y^2dy + z^2dz = \int_C \mathbf{F} \cdot d\mathbf{x} = \int \int_M [\nabla \times \mathbf{F}] \cdot d\mathbf{\sigma} = 0 \]

**Note:** In this case, since \( \nabla \times \mathbf{F} = 0 \), it wasn’t necessary to know details of \( M \) to get the answer; I drew it just for your understanding in case \( \nabla \times \mathbf{F} \) turned out nonzero.

(3) (Section 6.4, Problem 5) Evaluate the line integral \( \oint_C \mathbf{F} \cdot d\mathbf{x} \) using Stokes Theorem, where \( C \) is the rectangle with vertices at \( (0, 0, 4), (3, 0, 4), (3, 2, 4) \) and \( (0, 2, 4) \) and \( \mathbf{F} = \left( \frac{y}{z}, x^2y, x + z \right) \).
Solution: First we calculate
\[ \nabla \times \mathbf{F} = \left( \partial_y [x + z] - \partial_z [x^2 y], \partial_z \left[ \frac{y}{z} \right] - \partial_x [x + z], \partial_x [x^2 y] - \partial_y \left[ \frac{y}{z} \right] \right) \]
\[ = \left( 0, -\frac{y}{z^2} - 1, 2xy - \frac{1}{z} \right) \]

We notice on inspection that the (0, 0, 4), (3, 0, 4) and (3, 2, 4) and (0, 2, 4) are all on the plane \( z = 4 \). So, we choose \( M \) to be part of this plane that is bounded by the oriented rectangular boundary connecting the given points (see Fig.). We note that the projection on the \( x - y \) plane is the inside of the rectangle \( R \) with vertices at (0,0,0), (3, 0,0), (3, 2,0) and (0, 2,0). Since \( M \) is part of the plane \( z = 4 = h(x, y) \), we have \( h_x = 0 = h_y \). Note on \( M \), \( \nabla \times \mathbf{F} = \left( 0, -\frac{y}{16} - 1, 2xy - \frac{1}{4} \right) \). So, using Stokes’ Theorem

\[ \oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_M (\nabla \times \mathbf{F}) \cdot d\mathbf{σ} = \iint_R \left( 0, -\frac{y}{16} - 1, 2xy - \frac{1}{4} \right) \cdot (0,0,1) dA \]
\[ = \int_0^2 \int_0^3 \left[ 2xy - \frac{1}{4} \right] dxdy = \int_0^2 \left[ yx^2 - \frac{x}{4} \right]_{x=0}^{x=3} dy = \int_0^2 \left[ 9y - \frac{3}{4} \right] dy \]
\[ = \frac{9}{2} \left[ y^2 \right]_{y=0}^{y=2} - \frac{3}{2} = 18 - \frac{3}{2} = \frac{33}{2} \]

(4) (Section 6.4, Problem 6) Using Stokes’ Theorem, evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{x} \), where \( C \) is the triangle with verticies (0, 1, 0), (0, 1, 5)
and \((3, 1, 0)\) oriented by this ordering of the points and \(\mathbf{F} = (-xy, -xz, -yz)\).

**Figure 4.** Contour \(C\) along the triangle edges with vertices at \((0,1,0)\), \((0,1,5)\) and \((3,1,0)\) in the sense shown. Note \(C\) is the boundary of the triangle, whose projection in the \(x - z\) plane is an identical triangle. Note \(\mathbf{n} = \mathbf{j}\)

**Solution:** Note that

\[
\nabla \times \mathbf{F} = (\partial_y[-yz] - \partial_z[-xz], \partial_z[-xy] - \partial_x[-xz], \partial_x[-yz] - \partial_y[-xy])
\]

\[
= (-z + x, 0, -z + x)
\]

Surface \(M\) which has \(C\) as the boundary can be chosen to be the triangular part of the the plane \(M\) that contains the three given points \((0, 1, 0)\), \((0, 1, 5)\) and \((3, 1, 0)\), which is clearly on inspection \(y = 1\). If you could not guess this, note that \(M\) contains the vectors \(\mathbf{a} = (0,0,5)\) and \(\mathbf{b} = (3,0,0)\). Note \(\mathbf{a} \times \mathbf{b} = (0,15,0)\). This is the direction of the normal. So, equation of the plane is \(15(y-1) = 0\), i.e. \(y = 1\). So, normal \(\mathbf{n} = (0,1,0)\). We note \(\nabla \times \mathbf{F} \cdot \mathbf{n} = (-z+x, 0, -z+x) \cdot (0,1,0) = 0\).

Therefore using Stokes Theorem,

\[
\oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_M \mathbf{F} \cdot d\mathbf{\sigma} = 0
\]

**Note:** In this case there was no need to waste time to figure out projected region \(R\) in the \(x-z\) plane or even draw the figure as long as we recognize \(\mathbf{n} = (0,1,0) = \mathbf{j}\) is normal to \(\nabla \times \mathbf{F}\). I went through the details so you know what to do if you had a nonzero value.
(5) (Section 6.4, Problem 7) Using Stokes theorem calculate the line integral \( \oint_C \mathbf{F} \cdot d\mathbf{x} \), where \( C \) is the curve in which the surface \( z = x^2 - y^2 \) intersects the cylinder \( x^2 + y^2 = 1 \), oriented counterclockwise as viewed from the positive z-axis; \( \mathbf{F} = \left(-\frac{y}{2} + z, \frac{x}{2} + \frac{3}{2}z, -x - \frac{3}{2}y\right) \).

**Solution:** We will take \( M \) to be the part of the surface \( z = x^2 - y^2 = h(x, y) \) that is cut-off by the cylinder \( x^2 + y^2 = 1 \). I am not bothering to draw \( M \) since it is clear that the projection of \( M \) in the \( x - y \) plane is \( R \), which is the inside of the circle \( x^2 + y^2 \leq 1 \). Note \( h_x = 2x, h_y = -2y \). We now calculate

\[
\nabla \times \mathbf{F} = \left( \partial_y \left[ -x - \frac{3}{2}y \right] - \partial_z \left[ \frac{x}{2} + \frac{3}{2}z \right], \partial_z \left[ -\frac{y}{2} + z \right] - \partial_x \left[ -x - \frac{3}{2}y \right], \partial_x \left[ \frac{x}{2} + \frac{3}{2}z \right] - \partial_y \left[ -\frac{y}{2} + z \right] \right) = \left( -\frac{3}{2} - \frac{3}{2}, 1 + 1, \frac{1}{2} + \frac{1}{2} \right) = (-3, 2, 1)
\]

So,

\[
\oint_C \mathbf{F} \cdot d\mathbf{x} = \int \int_M (\nabla \times \mathbf{F}) \cdot d\mathbf{n} = \int \int_R (-3, 2, 1) \cdot (-2x, 2y, 1) \, dA
\]

\[
= \int \int_R (6x + 4y + 1) \, dA = \int_0^1 \int_0^{2\pi} [6r \cos \theta + 4r \sin \theta + 1] \, r \, d\theta \, dr
\]

\[
= \int_0^1 [6r^2 \sin \theta - 4r^2 \cos \theta + r\theta]_{\theta=0}^{\theta=2\pi} \, dr = \int_0^1 2\pi r \, dr = \pi \left[ r^2 \right]_0^1 = \pi
\]

(6) (Section 6.4, Problem 8) \( \oint_C \mathbf{F} \cdot d\mathbf{x} \), where \( C \) is the curve in which the plane \( x = 2 \) intersects the sphere \( x^2 + y^2 + z^2 = 16 \), oriented counter-clockwise as viewed from the positive x-axis and \( \mathbf{F} = (z - x^2, x - y^2, y - z^2) \).

**Solution:** We choose \( M \) to be the the part of the plane \( x = 2 \) that is cut-off by \( x^2 + y^2 + z^2 = 16 \). We note that where they intersect \( 2^2 + y^2 + z^2 = 16 \). So the projection of \( M \) on the \( y - z \) plane \( R \) (See Figure) is the inside of the circle \( y^2 + z^2 = 12 \). We now calculate \( \nabla \times \mathbf{F} \) on surface \( M \). We note from the statement about orientation of contour that \( \mathbf{n} = (1, 0, 0) \). We note

\[
\nabla \times \mathbf{F} = \left( \partial_y (y - z^2) - \partial_z (x - y^2), \partial_z (z - x^2) - \partial_x (y - z^2), \partial_x (x - y^2) - \partial_y (z - x^2) \right) = (1, 1, 1)
\]
We note that $\nabla \times \mathbf{F} \cdot \mathbf{n} = (1, 1, 1) \cdot (1, 0, 0) = 1$. Therefore, using Stokes Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int \int_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int \int_R 1 \, dA = \text{Area of } R = \pi(\sqrt{12})^2 = 12\pi$$

(7) (Section 6.4, Problem 9) Use Stokes Theorem to show that if $f$ has continuous second order partial derivative in a simply connected set $U$, then $\oint_C \nabla f \cdot d\mathbf{x} = 0$.

**Solution:** With given condition, Stokes Theorem is valid since condition on connected region implies we can find a surface $M$ whose boundary is $C$ and on which $\nabla f$ has continuous derivatives. So,

$$\oint_C \nabla f \cdot d\mathbf{x} = \int \int_M [\nabla \times (\nabla f)] \cdot \mathbf{n} \, d\sigma$$

We note that

$$\nabla \times (\nabla f) = \nabla \times (f_x, f_y, f_z) = (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{zy} - f_{yz}) = (0, 0, 0)$$

So, from above equation,

$$\oint_C \nabla f \cdot d\mathbf{x} = 0,$$

a result we proved in class in a different manner (exact differentials).

(8) (Section 6.4, Problem 10). Use Stokes Theorem to show that if $M$ is a piecewise smooth closed surface, then for any continuously differentiable vector field $F$ whose domain contains $M$ (interior included), $\int \int_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 0$. 

**Figure 5.** Contour $C$ at the intersection of $x = 2$ plane with $x^2 + y^2 + z^2 = 16$ oriented counter-clockwise from top. $M$ is part of the the plane $x = 2$ bounded by $C$. Note $\mathbf{n} = \mathbf{i} = (1, 0, 0)$ and the unconventional axes choice for clarity purposes.
Solution: We denote the interior of $M$ by the solid region $S$. The conditions on $M$ and $\mathbf{F}$ imply that we can use divergence Theorem on $\nabla \times \mathbf{F}$ to obtain

$$\int\int_{M} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int\int\int_{S} \nabla \cdot (\nabla \times \mathbf{F}) dV$$

But

$$\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot (\partial_{y}F_{3} - \partial_{z}F_{2}, \partial_{z}F_{1} - \partial_{x}F_{3}, \partial_{x}F_{2} - \partial_{y}F_{1})$$

$$= \partial_{yx}F_{3} - \partial_{zx}F_{2} + \partial_{zy}F_{1} - \partial_{xy}F_{3} + \partial_{xz}F_{2} - \partial_{yz}F_{1} = 0$$

So,

$$\int\int_{M} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = 0$$

(9) From the given figure (without any formula) for the vector field $\mathbf{F}$ determine whether or not you can ascertain whether $\nabla \times \mathbf{F}$ is nonzero. If the answer is nonzero sketch the direction of curl.

![Figure 6. Sketch of vector field $\mathbf{F}$](image_url)

Solution: Clearly, from the figure, we can choose a closed loop $C$ around the $z$-axis so that it aligns with the vector field $\mathbf{F}$. On that loop $C \mathbf{F}$ and tangent vector $d\mathbf{x}$ are in the same direction and so from definition of dot product $\mathbf{F} \cdot d\mathbf{x} > 0$. Therefore, $\oint_{C} \mathbf{F} \cdot d\mathbf{x} > 0$.

Since, we know from Stokes theorem $[\nabla \times \mathbf{F}] \cdot \mathbf{k}$ is the loop integral per unit area as the loop shrinks to zero size with loop plane having normal $\mathbf{n} = \mathbf{k}$ (along the positive $z$-axis) we can be assured that at least the $\mathbf{k}$ component of $\nabla \times \mathbf{F} \neq 0$ for any point on the $z$-axis. Thus $\nabla \times \mathbf{F} \neq 0$. Clearly the maximum value of loop integral for a point on the $z$-axis is when the loop
normal is along \( k \). So direction of curl is roughly along \( k \) (Can’t be more precise since the figure has some imprecision.)

(10) (From earlier section on divergence) Sketch a vector field \( F \) which has obviously nonzero divergence (Hint: You may think of what divergence means.

\[
\text{Figure 7. Sketch of vector field } F \text{ with nonzero divergence at a point}
\]

**Solution:** We sketch as shown in the figure above. Clearly, we can imagine a small closed surface \( M \) with normal vector \( n \) aligned along the \( F \), which will ensure \( F \cdot n > 0 \) Then, \( \int_{\partial M} F \cdot n \, d\sigma > 0 \). Since outwards flux per unit volume in the limit of a small volume around a point is the divergence \( \nabla \cdot F \) at that point, it follows that there is at least a point (namely where the tail of the arrows meet), where where \( \nabla \cdot F > 0 \) and hence \( \nabla \cdot F \neq 0 \) identically.

(11) Suppose \( F = xe^x i + \sqrt{x^2 + y^2 + z^2} j + xy k \). Describe \( F \) using cylindrical basis \( \{ e_r, e_\theta, k \} \).

**Solution:** We have, using relation between \( x, y \) and \( r, \theta \),

\[
F = r \cos \theta e^{r \cos \theta} i + \sqrt{r^2 + z^2} j + r^2 \cos \theta \sin \theta k
\]

\[
= r \cos \theta e^{r \cos \theta} [e_r \cos \theta - e_\theta \sin \theta] + \sqrt{r^2 + z^2} [e_r \sin \theta + e_\theta \cos \theta] + r^2 \cos \theta \sin \theta k
\]

\[
= \left\{ r \cos^2 \theta e^{r \cos \theta} + \sqrt{r^2 + z^2} \sin \theta \right\} e_r + \left\{ -r \cos \theta \sin \theta e^{r \cos \theta} + \sqrt{r^2 + z^2} \cos \theta \right\} e_\theta
\]

\[
+ r^2 \cos \theta \sin \theta k
\]
(12) If \( f(x, y, z) = \sqrt{x^2 + y^2} e^z \) calculate \( \nabla f \) in using cylindrical basis functions.

**Solution:** Note that \( f = r e^z \). So, in cylindrical coordinates,

\[
\nabla f = \frac{\partial}{\partial r} [re^z] e_r + \frac{1}{r} \frac{\partial}{\partial \theta} [re^z] e_\theta + \frac{\partial}{\partial z} [re^z] e_z = e^z e_r + re^z k
\]

(13) If \( \mathbf{F} = \frac{x}{|x|^3} \) calculate \( \nabla \cdot \mathbf{F} \) using cylindrical representation.

**Solution:** Note that

\[
\mathbf{F} = \frac{x}{(x^2 + y^2 + z^2)^{5/2}} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{5/2}} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{5/2}} \mathbf{k}
\]

\[
= \frac{r \cos \theta}{(r^2 + z^2)^{5/2}} \mathbf{i} + \frac{r \sin \theta}{(r^2 + z^2)^{5/2}} \mathbf{j} + \frac{z}{(r^2 + z^2)^{5/2}} \mathbf{k}
\]

Recall in cylindrical coordinates,

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}
\]

\[
= \frac{\partial}{\partial r} \left[ r(r^2 + z^2)^{-5/2} \right] + (r^2 + z^2)^{-5/2} + \frac{\partial}{\partial z} \left[ z(r^2 + z^2)^{-5/2} \right]
\]

\[
= (r^2 + z^2)^{-5/2} - 5r^2(r^2 + z^2)^{-7/2} + (r^2 + z^2)^{-5/2} + (r^2 + z^2)^{-5/2} - 5z^2(r^2 + z^2)^{-7/2}
\]

\[
= 3(r^2 + z^2)^{-5/2} - 5(r^2 + z^2)(r^2 + z^2)^{-7/2} = -2(r^2 + z^2)^{-5/2} = -\frac{2}{|x|^5}
\]

(14) Show that \( r^n \cos(n\theta) \) and \( r^{-n} \cos n\theta \) each solve \( \nabla^2 f = 0 \) for any \( n \).

**Solution:** We recall that in cylindrical coordinates

\[
\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\]

Now if \( f = r^n \cos \theta \), then

\[
\nabla^2 f = \frac{\partial^2}{\partial r^2} [r^n \cos n\theta] + \frac{1}{r} \frac{\partial}{\partial r} [r^n \cos n\theta] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} [r^n \cos n\theta]
\]

\[
= n(n-1)r^{n-2} \cos n\theta + nr^{n-2} \cos n\theta - n^2r^{n-2} \cos n\theta = [n^2 - n + n - n^2]r^n \cos n\theta = 0
\]

So, \( \nabla^2 [r^n \cos n\theta] = 0 \) Replacing \( n \) by \( -n \) has no effect in the above calculation, except that if \( -n < 0 \), we have to exclude \( r = 0 \) from the calculation. So, if \( n > 0 \), \( \nabla^2 [r^{-n} \cos n\theta] = 0 \), except for \( r = 0 \).