1a. In terms of a double integral with appropriate limits determine an expression for the surface area of the part of the surface \( z = x^2 + y^2 \) that is inside the cylinder \( x^2 + y^2 = 4 \).

**Solution:** Note on the surface \( M \), \( z = h(x, y) = x^2 + y^2 \) and so \( h_x = 2x \) and \( h_y = 2y \) and so \( \sqrt{1 + h_x^2 + h_y^2} = \sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2} \).

Further the projection \( R \) of the surface in the \( x-y \) plane is clearly the interior of the circle \( x^2 + y^2 \leq 4 \), i.e. \( r \leq 2 \). So, the surface area

\[
A = \int \int_M d\sigma = \int \int_R \sqrt{1 + h_x^2 + h_y^2} dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r dr d\theta
\]

1b. With appropriate changes of variable determine the area integral \( \int \int_R \sin(2x) dA \), where \( R \) is the region bounded by \( x-y = 0, x-y = 1, x+y = 0, x+y = 2 \).

**Solution:** Choose \( s = x+y, t = x-y \). Note region \( R \) corresponds to \( \{(s,t) : 0 \leq s \leq 2, 0 \leq t \leq 1 \} \). Also, note \( 2x = s+t \) and \( \frac{\partial(s,t)}{\partial(x,y)} = s_y t_x - s_x t_y = -2 \). So, Jacobian \( \frac{\partial(x,y)}{\partial(s,t)} = -\frac{1}{2} \).

\[
\int \int_R \sin(2x) dA = \int_0^1 \int_0^2 \sin(s+t) \frac{1}{2} ds dt = \frac{1}{2} \int_0^1 [-\cos(s+t)]_{s=0}^{s=2} dt = \frac{1}{2} \int_0^1 [\cos t - \cos(2+t)] dt = \frac{1}{2} [\sin t - \sin(2+t)]_0^1 = \frac{1}{2} [\sin 1 - \sin 3 + \sin 2]
\]
2a. Determine
\[
\int_C x\,dx + x\,dy + y\,dz
\]
where \(C\) is a straight line from \((0, 2, 3)\) to \((-1, 3, 0)\).

**Solution:** Parametric representation of \(C\) is clearly \((0, 2, 3) + t(-1, 3-2, 0-3) = (-t, 2+t, 3-3t)\). Therefore, \(dx = -dt\), \(dy = dt\), \(dz = -3dt\).

So,
\[
\int_C x\,dx + x\,dy + y\,dz = \int_0^1 \left[ -t(-dt) + (t)dt + (2+t)(-3dt) \right] = \int_0^1 (2+t)dt
\]
\[
= \left[ -6t + \frac{3}{2} t^2 \right]_0^1 = -6 - \frac{3}{2}
\]

2b. Determine if the answer in (1b) will generally depend on the path connecting the two given points.

**Solution:** We calculate \(\nabla \times (x, x, y) = (\partial_y y - \partial_z x, \partial_z x - \partial_y y, \partial_x x - \partial_y x) = (1, 0, 1) \neq 0\). Hence \(F = (x, x, y)\) is not conservative and hence the answer generally depends on the path connecting the two given points.
3a. Determine if $F = (ye^{xy} - 2x)i + (xe^{xy} + 2y)j$ is conservative. If so, determine the scalar potential $f$ so that $F = \nabla f$.

**Solution** Note $\partial_x[xe^{xy} + 2y] = e^{xy} + xy e^{xy} = \partial_y[ye^{xy} - 2x]$. So, $F$ is conservative. We have therefore,

$$(ye^{xy} - 2x, xe^{xy} + 2y) = F = \nabla f = (f_x, f_y)$$

So, partial integration in $x$ gives $f = e^{xy} - x^2 + g(y)$. Substituting into second expression, we have

$$xe^{xy} + g'(y) = xe^{xy} + 2y$$

So, $g(y) = y^2 + C$. Therefore, scalar potential

$$f(x, y) = e^{xy} - x^2 + y^2 + C$$

3b. Determine the upwards flux of the vector field $F = zk$ across the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$.

**Solution:** Though the surface $M$ of the given hemisphere is open, we note that if we took $\partial S$ to be the closed hemisphere with a bottom lid that coincided with the plane $z = 0$, then the flux integral contribution from the bottom lid is zero since $F \cdot n = zk \cdot n = 0$ on $z = 0$. So, since $\nabla \cdot F = 1$,

$$\int \int_M F \cdot nd\sigma = \int \int_{\partial S} F \cdot nd\sigma = \int \int \nabla \cdot F dV = 1 \int \int \int dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \frac{2}{3} \pi$$
4a. Determine the outwards flux of the vector field

\[ \mathbf{F} = z \sin y \mathbf{i} + e^x \mathbf{j} + xy \mathbf{k} \]

through the surface \( \partial S \) of the solid bounded by \( z = 0 \) and \( z = 4 - \sqrt{x^2 + y^2} \).

**Solution:** We note that we now have a closed surface \( \partial S \) and that

\[ \nabla \cdot \mathbf{F} = \partial_x[z \sin y] + \partial_y[e^x] + \partial_z[xy] = 0. \]

Hence, using divergence theorem, which is applicable since \( \mathbf{F} \) is smooth on inspection, the total outwards flux through \( \partial S \) is zero.

4b. Calculate \( \oint_{C} \mathbf{F} \cdot d\mathbf{x} \) where

\[ \mathbf{F} = 2z \mathbf{i} + 6x \mathbf{j} - 3y \mathbf{k} \]

and \( C \) is the counter-clockwise closed path \( x^2 + y^2 = 1, z = 0 \).

**Solution:** We take parametric representation \( x = \cos t, y = \sin t, z = 0 \) for the path. Then \( dx = -\sin t \, dt, dy = \cos t \, dt, dz = 0 \)

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{x} = \int_{0}^{2\pi} 6 \cos^2 t \, dt = 3 \int_{0}^{2\pi} [1 + \cos 2t] \, dt = 6\pi \]
5a. For the vector field $\mathbf{F} = 3xy\mathbf{i} + y^2\mathbf{j} - x^2y^4\mathbf{k}$, calculate $\oint_C \mathbf{F} \cdot d\mathbf{x}$, where $C$ is the counter-clockwise closed path $x^2 + y^2 = 1, z = 0$.

**Solution:** Note that on $C$, if we use $x = \cos t, y = \sin t$ and $z = 0$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} 3[\cos t \sin t][-\sin t dt] + \sin^2 t[\cos t dt] = \left[ -\sin^3 t + \frac{1}{3} \sin^3 t \right]_0^{2\pi} = 0$$

5b. For the vector field $\mathbf{F}$ in (4a), determine

$$\int \int_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA$$

where $M$ is the surface of the part of the ellipsoid $x^2 + y^2 + 2z^2 = 1$ with $z \geq 0$ and $\mathbf{n}$ is taken to have a positive $\mathbf{k}$ component.

**Solution:** Since surface $M$ is smooth and the boundary $C$ defined in part a. is its positively oriented boundary and the vector field $\mathbf{F}$ is clearly smooth, it follows that Stokes Theorem is applicable and therefore,

$$\int \int_M [\nabla \times \mathbf{F}] \cdot \mathbf{n} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{x} = 0$$

from calculation in the first part.