Solution to Section 5.4, assigned April 23

(1) (Section 5.4, Problem 10) Evaluate the iterated triple integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xydzdydx$$

Solution:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xydzdydx = \int_0^2 \int_0^{\sqrt{4-x^2}} [xyz]_{z=0}^{z=\sqrt{4-x^2=y^2}} dydx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} xy\sqrt{4-x^2-y^2}dydx = \int_0^2 \left[-\frac{x}{3}(4-x^2-y^2)^{3/2}\right]_0^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 \frac{x}{3}(4-x^2)^{3/2}dx = \left[-\frac{1}{15}\left(4-x^2\right)^{5/2}\right]_0^2 = \frac{32}{15}$$

(2) (Problem 5.4, Problem 14) Evaluate the volume integral (triple integral) of $f(x,y,z) = x^2$ over S, where S is the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.
Solution:

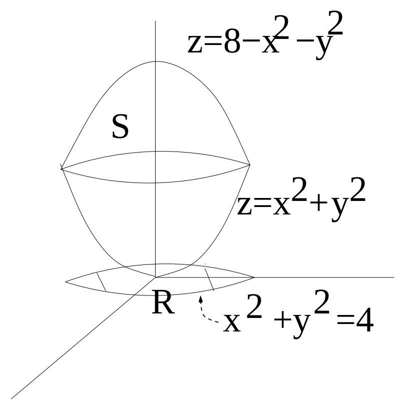

FIGURE 1. Region S bounded above by paraboloid $z = 8-x^2-y^2$ and below by paraboloid $z = x^2+y^2$. Surfaces intersect on the curve $x^2 + y^2 = 4 = z$. So boundary of the projected region R in the $x-y$ plane is $x^2 + y^2 = 4$.

Where the two surfaces intersect $z = x^2 + y^2 = 8 - x^2 - y^2$. So, $2x^2 + 2y^2 = 8$ or $x^2 + y^2 = 4 = z$, this is the curve at the intersection of the two surfaces. Therefore, the boundary of projected region R in the $x-y$ plane is given by the circle $x^2 + y^2 = 4$. So R can be treated as a y simple region in the

$x-y$ plane, with upper and lower curves $y = \pm\sqrt{4-x^2}$ for $-2 \le x \le 2$. Therefore,

$$\int\int\int_S x^2 dV = \left\{\int\int\right\}_R \left[\int_{x^2+y^2}^{8-x^2-y^2} x^2 dz\right] dA$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[x^2 z\right]_{z=x^2+y^2}^{z=8-x^2-y^2} dydx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2\left(8-2x^2-2y^2\right) dydx$$

$$= \int_{-2}^{2} x^2 \left[8y - 2x^2 y - \frac{2}{3}y^3\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} x^2 \left(16\sqrt{4-x^2} - 4x^2\sqrt{4-x^2} - \frac{4}{3}\left(4-x^2\right)^{3/2}\right) dx$$

Substituting $x = 2\sin\theta$ and noting that $dx = 2\cos\theta d\theta$ we get

$$\int\int\int_S x^2 dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\left(256\cos\theta - 256\sin^2\theta\cos\theta - \frac{256}{3}\cos^3\theta\right)\cos\theta d\theta$$

$$= \int_0^{\pi/2} d\theta\left(512\cos^2\theta\sin^2\theta - 512\sin^4\theta\cos^2\theta - \frac{512}{3}\cos^4\theta\sin^2\theta\right)$$

Using $512\cos^2\theta\sin^2\theta = 128\sin^2(2\theta) = 64\left(1-\cos[4\theta]\right)$,

$$-512\sin^4\theta\cos^2\theta = -128\sin^2(2\theta)\sin^2\theta = -32\left[1-\cos 4\theta\right]\left[1-\cos 2\theta\right]$$
$$= -32 + 32\cos 2\theta + 32\cos 4\theta - 16\cos 2\theta - 16\cos 6\theta$$

$$-\frac{512}{3}\cos^4\theta\sin^2\theta = -\frac{128}{3}\sin^2(2\theta)\cos^2\theta = -\frac{32}{3}\left[1-\cos 4\theta\right]\left[1+\cos 2\theta\right]$$
$$= -\frac{32}{3}\left[1 + \cos 2\theta - \cos 4\theta - \frac{1}{2}\cos 2\theta - \frac{1}{2}\cos 6\theta\right]$$

Since the integral of $\cos[2m\theta]$ for $m = 1, 2, 3$ is a multiple of $\sin[2m\theta]$ which is zero at $\theta = \pi/2$, it follows that

$$\int\int\int_S x^2 dV = \left(64 - 32 - \frac{32}{3}\right)\left[\theta\right]_0^{\frac{\pi}{2}} = \frac{32}{3}\pi$$

(3) (Section 5.4, Problem 18) Find the volume of the indicated solid region S inside the cylinders $x^2+y^2 = a^2$ and $x^2+z^2 = a^2$. **Solution:** Consider only the part of S that lies in the region $x \ge 0$, $y \ge 0$, $z \ge 0$ From symmetry of the region under the transformation $x \to -x$, $y \to -y$ and $z \to -z$, it follows that the volume of this region S_1 is $\frac{V}{8}$, where V is the volume of S. We treat S_1 as an x-simple region in 3-D.

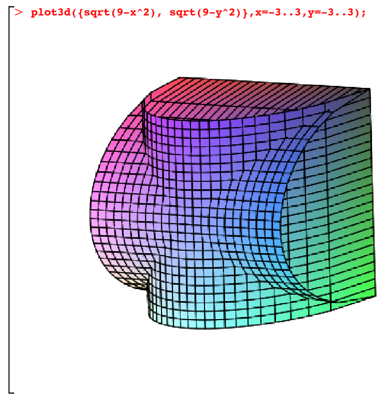

FIGURE 2. Part of the region S bounded by $x^2+z^2=a^2$ and $x^2+y^2=a^2$ for $x \geq 0$

Note that the projection of region S_1 on the $y-z$ plane, call it R is a a square $0 \leq y \leq a$, $0 \leq z \leq a$. We break up R into two region $R_1 = \{(y,z) : a \geq y \geq z \geq 0, \}$ and $R_2 = \{(y,z) : a \geq z > y \geq 0\}$. In region $(y,z) \in R_1$, x-ranges from $x=0$ to $x=\sqrt{a^2-y^2}$ (since this is smaller than $\sqrt{a^2-z^2}$. In region $(y,z) \in R_1$, x-ranges from $x=0$ to $x=\sqrt{a^2-z^2}$ (since this is smaller than $\sqrt{a^2-y^2}$. So, it follows that the total volume of S_1 is

$$\frac{V}{8} = \int_{R_1} \left[\int_0^{\sqrt{a^2-y^2}} dx \right] dA + \int_{R_2} \left[\int_0^{\sqrt{a^2-z^2}} dx \right] dA$$

$$= \int_0^a \int_0^y \sqrt{a^2-y^2} dz dy + \int_0^a \int_0^z \sqrt{a^2-z^2} dy dz = 2 \int_0^a y\sqrt{a^2-y^2} dy$$

$$= -\frac{2}{3} \left[(a^2-y^2)^{3/2} \right]_0^a = \frac{2}{3} a^3$$

Therefore, volume of S is $V = \frac{16}{3} a^3$.

(4) (Section 5.4, Problem 24). Find the centroid of the given solid bounded by the paraboloids $z = 1+x^2+y^2$ and $z = 5-x^2-y^2$ with density proportional to the distnace from the $z = 5$ plane.
Solution: From the problem statement, density $\rho = k|z-5| = k(5-z)$ since region is below plane $z = 5$. The plot of the region S between the two paraboloids is similar to (Secion 5.4, Problem 14) we have solved above, whose projection R in the $x-y$ plane is bounded by the curve given by $1 + x^2 + y^2 = 5 - x^2 - y^2$, or $x^2 + y^2 = 2$. So, we have mass

$$M = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{1+x^2+y^2}^{5-x^2-y^2} k(5-z)dzdydx$$

$$= k \int_{-\sqrt{2}}^{\sqrt{2}} \int_{0}^{\sqrt{2-x^2}} \left[16 - 8x^2 - 8y^2\right] dydx = k \int_{-\sqrt{2}}^{\sqrt{2}} \left[16\sqrt{2-x^2} - 8x^2\sqrt{2-x^2} - \frac{8}{3}(2-x^2)^{3/2}\right] dx$$

$$= \frac{32}{3}k \int_{0}^{\sqrt{2}} (2-x^2)^{3/2}dx = \frac{128}{3}k \int_{0}^{\pi/2} \cos^4\theta d\theta = \frac{32}{3}k \int_{0}^{\pi/2} \left[1 + \frac{1}{2} + \frac{1}{2}\cos(4\theta) + 2\cos(2\theta)\right] d\theta$$

$$= 8\pi k$$

Now, from symmetry of the shape, it follows that $x_c = 0 = y_c$. So, we only need to calculate

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{1+x^2+y^2}^{5-x^2-y^2} kz(5-z)dzdydx$$

$$= 2k \int_{-\sqrt{2}}^{\sqrt{2}} \int_{0}^{\sqrt{2-x^2}} \left[\frac{5}{2}z^2 - \frac{z^3}{3}\right]_{z=1+x^2+y^2}^{z=5-x^2-y^2} dydx$$

$$= 2k \int_{-\sqrt{2}}^{\sqrt{2}} \int_{0}^{\sqrt{2-x^2}} \left[-4x^2 - 4y^2 - 8x^2y^2 + 2x^4y^2 + 2x^2y^4 + \frac{56}{3} + \frac{2}{3}x^6 + \frac{2}{3}y^6 - 4x^4 - 4y^4\right] dydx$$

$$= 4k \int_{0}^{\sqrt{2}} \left[\frac{1424}{105}\sqrt{2-x^2} - \frac{152}{35}x^2\sqrt{2-x^2} - \frac{64}{35}x^4\sqrt{2-x^2} + \frac{32}{105}x^6\sqrt{2-x^2}\right] dx$$

$$= 4k \left[\frac{16}{3}x(2-x^2)^{1/2} + \frac{32}{3}\arcsin\left(\frac{x}{2^{1/2}}\right) + \frac{152}{105}x(2-x^2)^{3/2} + \frac{76}{315}x^3(2-x^2)^{3/2} - \frac{4}{105}x^5(2-x^2)^{3/2}\right]_0^{\sqrt{2}} = \frac{64\pi}{3}k$$

So, $z_c = \frac{64\pi k}{3(8\pi k)} = \frac{8}{3}$ and $\mathbf{x}_c = \left(0, 0, \frac{8}{3}\right)$.

(5) (Section 5.4, Problem 27) Reverse the order of integration appropriate for a z-simple and x-simple regions.

$$\int_0^2 \int_0^{\sqrt{1-z^2/4}} \int_0^{3\sqrt{1-x^2-z^2/4}} f(x,y,z)dydxdz$$

Solution: Since y ranges from 0 to $y = 3\sqrt{1-x^2-z^4/4}$, we have the upper surface $\frac{y^2}{9} + x^2 + \frac{z^2}{4} = 1$, which is an ellipsoid. We also note that the projected region R in the $x-z$ plane has goes between $x = 0$ and $x = \sqrt{1-z^2/4}$, the latter being the boundary of an ellipse, while z ranges from 0 to 2. Therefore, it is clear that the region S is the first octant of an ellipsoid bounded by $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

Treating S as a z-simple region, we have lower surface $z = 0$ and upper-surface $z = 2\sqrt{1-x^2-\frac{y^2}{9}}$. The projected region in the $x-y$ is the the inside of the ellipse $x^2 + \frac{y^2}{9} = 1$ in the first quadrant, which may be described as a y-simple region in the 2-D $x-y$ plane:

$$\left\{(x,y) : 0 \le y \le 3\sqrt{1-x^2}, 0 \le x \le 1\right\}$$

So, the integral above is the same as

$$\int_0^1 \int_0^{3\sqrt{1-x^2}} \int_0^{2\sqrt{1-x^2-\frac{y^2}{9}}} f(x,y,z)dzdydx$$

Treating S as a x simple region, we have for fixed $y-z$, x going from 0 to $\sqrt{1-\frac{y^2}{9}-\frac{z^2}{4}}$. The projected region in the $y-z$ plane can be described as a z-simple region in the $y-z$ plane and described by

$$\left\{(y,z) : 0 \le z \le 2\sqrt{1-\frac{y^2}{9}}, 0 \le y \le 3\right\}$$

So, the above integral is the same as

$$\int_0^3 \int_0^{2\sqrt{1-\frac{y^2}{9}}} \int_0^{\sqrt{1-\frac{y^2}{9}-\frac{z^2}{4}}} f(x,y,z)dxdzdy$$

(6) (Section 5.4, Problem 30) Using Theorem 5.4.3, determine whether the integral $\int\int\int_S zdV$ is positive, negative or 0, where S is the solid bounded by the paraboloid $z = -x^2 - y^2$ and the plane $z = -4$.

Solution: Note from the description of the region that $f(x, y, z) = z < 0$ in S. Therefore, from theorem 4.5.3, $\int\int\int_S z dV < 0$.