Finals, Math 6451, Due: Friday, April 28th, 5 p.m.

Instructions: No collaboration or discussion. Ask instructor for any clarifications

1. Determine solution to the PDE as explicitly as possible:

$$u_{x_1} + 2u_{x_2} + (2x_1 - x_2)u = x_1 x_2, \quad \text{with} \quad u(0, x_2) = e^{x_2}$$

2. For given continuous function $a(\mathbf{x}, t)$, prove an appropriate maximum principle for $u(\mathbf{x}, t)$ satisfying:

$$u_t + \mathbf{a} \cdot \nabla u = \Delta u - u^2 \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n \ , \ t > 0$$

where Ω is a bounded domain, with $u(\mathbf{x}, 0) = \phi(\mathbf{x})$, for $\phi \in C^0(\partial\Omega)$ and $u = 0$ on $\partial\Omega$. Use this to show that a solution u, if it exists, cannot blow up positively.

3. With appropriate assumptions on initial condition ϕ, ψ in the following, use energy method to prove uniqueness and continous dependence on initial condition of classical solution to the initial value problem for the damped wave equation ($\epsilon > 0$):

$$u_{tt} + \epsilon u_t = \Delta u \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n, \ t > 0 \text{, with } u(\mathbf{x}, 0) = \phi(\mathbf{x}), \ u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \ , \ u(\mathbf{x}, 0) = 0 \text{ on } \partial\Omega$$

4. With appropriate assumption on initial condition, prove that the following nonlinear PDE has unique classical solution for all time

$$u_t + u + (1 + u^2)u_x - u_{xx} = 0 \ , \quad \text{for } x \in \mathbb{R}, \ t > 0 \quad , \quad \text{with } u(x, 0) = F(x)$$

(**Hint:** A transformation to get rid of the undifferentiated u term might be helpful in using maximum principle to go from local existence to global existence.)