Homework Set 1: Math 6451, Due: Friday, January 27th

1. Calculate and plot the characteristic curves of

$$yu_x + xu_y = 0$$

where $(x, y) \in \mathbb{R}^2$. Afterwards, derive the general solution $u(x, y)$ under the additional assumption that u is even in x and y. On which of the coordinate axes do we have to impose initial conditions to make the solution unique.

Hint: You may want to describe the characteristic curves in the x-y variables.

2. Solve

$$u_x + u_y + u = e^{x+2y}$$

with $u(x, 0) = 0$.

3. Use the method of characteristics to find representation for solution to

$$u_t + u^2 u_x = u \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad u(x, 0) = e^{-x}$$

4. By direct verification show that for any integer n, $u(x, t) = e^{-n} e^{\kappa n^2 t} \sin nx$ is a solution to the backwards heat equation

$$u_t = -\kappa u_{xx} \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \quad \text{with} \quad u(x, 0) = e^{-n} \sin nx$$

Use this to prove that the solution is unstable to variations in initial conditions and therefore the problem is *not* well-posed in the $\|.\|_\infty$ (sup norm in x).

5. A function $u(x, t)$, defined for $(x, t) \in \mathbb{R}^2$, is said to be a weak solution of the linear second order wave equation: $u_{tt} - c^2 u_{xx} = 0$ if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) \left[\phi_{tt}(x, t) - c^2 \phi_{xx}(x, t) \right] dx dt = 0$$

for all test functions $\phi(x, t)$ with compact support, *i.e.* for any smooth function $\phi(x, t)$ that is defined for $(x, t) \in \mathbb{R}^2$ and that vanishes outside a bounded region of $\mathbb{R}^2$. Verify that

$$u(x, t) = f(x - ct) + g(x + ct)$$

is a weak solution for any continuous functions f and g.

Hint: You may think of transforming independent variables from $(x, t) \to (\xi, \eta)$, where $\xi = x - ct$, $\eta = x + ct$.