Homework Set 2: Math 6451, Due Friday, February 10

1. Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

for $x \in \mathbb{R}$ with $c \neq 0$. Assume $\phi \in \mathbf{C}^2$ and $\psi \in \mathbf{C}^1$ have compact support (*i.e.* there is a number $R > 0$ such that $\phi(x) = \psi(x) = 0$ for $|x| > R$). Denote d'Alembert's solution of (1) by $u(x,t)$. Using d'Alembert's solution, show that the solutions to (1) depend in a stable fashion on the initial data ϕ and ψ. More precisely: if u_1 and u_2 satisfy (1) with initial data (ϕ_1, ψ_1) and (ϕ_2, ψ_2), respectively, show that

$$\sup_{x\in\mathbb{R}} |u_1(x,t) - u_2(x,t)| \leq C(t) \left[\sup_{x\in\mathbb{R}} |\phi_1(x) - \phi_2(x)| + \sup_{x\in\mathbb{R}} |\psi_1(x) - \psi_2(x)| \right]$$

for some constant $C(t)$ that may depend on t, but not on the initial data. What is a good estimate for $C(t)$? Can you find similar estimates for $\partial_x(u_1 - u_2)$ and $\partial_{xx}(u_1 - u_2)$?

2. Find an integral representation of the solution to advection-diffusion equation

$$u_t + a u_x = u + \kappa u_{xx} \quad , -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \phi(x) \quad , -\infty < x < \infty$$

where a and κ are constants.

Hint: Transform independent variable $(x,t) \to (x - at, t)$, and replace u by $v = e^{-t}u$ as the dependent variable.

3. Consider the equation

$$u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0$$

$$u(0,t) = 0 = u(1,t) \qquad t > 0$$

$$u(x,0) = 4x(1-x) \qquad 0 < x < 1$$

Show that:

1) $0 < u(x,t) < 1$ for all $0 < x < 1$ and $t > 0$.

2) $u(x,t) = u(1-x,t)$ for all $0 < x < 1$ and $t > 0$.

3) The energy $\int_0^1 u^2(x,t)dx$ decreases strictly with t for $t > 0$.

Hint: In some of these, you may need to use the strong maximum principle that states that maximum cannot be attained in the interior points, except for trivial solutions where u is a constant.

4. Prove the comparison principle: If $u(x,t)$ and $v(x,t)$ are two solutions of the heat equation for $0 < x < l$ and $t \geq 0$ such that $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for all $0 \leq x \leq l$ and $t > 0$.