Homework Set 3: Math 6451, Due: Wednesday, February 22nd

1. Find a representation for the solution to wave equation on a half-line

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad x \in \mathbb{R}^+, t \in \mathbb{R}$$

with initial and boundary conditions:

$$u(x,0) = \sin x \quad ; \quad u_t(x,0) = 0 \quad ; \quad u_x(0,t) = 1$$

2. Find a solution to the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

for $x \in \mathbb{R}$ with $c \neq 0$. Assume $\phi \in \mathbf{C}^2$, $\psi \in \mathbf{C}^1$, and $f \in \mathbf{C}^0$ are given bounded functions.

Hint: Use Duhammel's principle, which for ODEs, says the following: If $v(t;\tau)$ is the solution to

$$\frac{dv}{dt} = Av, \quad v(\tau;\tau) = f(\tau) \quad \text{then} \quad u(t) = \int_0^t v(t;\tau)d\tau$$

is a solution to $\frac{du}{dt} = Au + f(t)$, $u(0) = 0$. Now, you want to generalize for a differential operator A depending on $\mathbf{x}$. Recall, we used it in class for heat equation.

3. Prove the weak maximum principle for Laplace's equation. Thus, assume that $\mathcal{D}$ is an open and bounded subset of $\mathbb{R}^n$ with boundary ∂D. Assume also that $u(\mathbf{x})$ is a solution to Laplace's equation $\Delta u = 0$ in $\mathcal{D}$ and that u is continuous on $\bar{\mathcal{D}}$, twice differentiable in D. Show that

$$\sup_{\bar{D}} u = \sup_{\partial D} u$$

4. a. Prove that if there exists a solution of the Neumann problem

$$\Delta u = f \quad \text{for} \quad x \in \mathcal{D} \subset \mathbb{R}^n, \quad \frac{\partial u}{\partial n} = h(x) \quad \text{for} \quad x \in \partial D,$$

then it is unique up to adding an arbitrary constant.

b. Consider the Robin problem

$$\Delta u = f \quad \text{for} \quad x \in \mathcal{D}, \quad \frac{\partial u}{\partial n} + a(x)u = h(x) \quad \text{for} \quad x \in \partial D$$

with $a(x) > 0$. Show that it has a unique solution.