Homework Set 5: Math 6451, Due Friday, March 25

1. Using representation for Green's function of 3-D wave equation, and combining it with representation of solution to homogeneous wave equation, derive formula for solution to in-homogeneous wave equation for $\mathbf{x} \in \mathbb{R}^3$

$$u_{tt} = \Delta u + f(\mathbf{x}, t) \ , \qquad , u(\mathbf{x}, 0) = \phi(\mathbf{x}) \ , u_t(\mathbf{x}, 0) = \psi(\mathbf{x})$$

State any reasonable restriction you need for f, ϕ and ψ.

2. Determine the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for Dirichlet condition for Laplace's equation in 3-D dimensions in the hemispherical domain:

$$\Omega = \{\mathbf{x} : |\mathbf{x}| < 1, \quad x_3 > 0\}$$

Use this to determine an integral expression for $u(r, \theta, \phi)$ in spherical polar-coordinates satisfying

$$\Delta u = 0 \ \ \text{in}\Omega \ \ , \text{with} \ \ u = 0 \ \ \text{for } \theta = 0, \ \ \text{and} \ \ u(1, \theta, \phi) = \psi(\theta, \phi) \ \ \text{for } \theta \in [0, \frac{\pi}{2}], \phi \in [0, 2\pi]$$

Hint: Better not to use spherical polar coordinates until after you get an integral representation.

3. If $\mathcal{S}(\mathbf{x}, t)$ is the source solution to the heat equation given by

$$\mathcal{S}(\mathbf{x}, t) = \left(\frac{1}{4\pi\kappa t} \right)^{n/2} \ \exp \left[-\frac{|\mathbf{x}|^2}{4\kappa t} \right],$$

then show that

$$R(\mathbf{x}, t; \mathbf{x}_0, t_0) = \mathcal{S}(\mathbf{x} - \mathbf{x}_0, t - t_0) \ \ \text{for } t > t_0 \ \ \text{and} \ \ R = 0 \ \ \text{for } t < t_0 \ \ \mathbf{x} \in \mathbb{R}^n$$

satisfies

$$R_t - \kappa \Delta R = \delta(\mathbf{x} - \mathbf{x}_0) \delta(t - t_0)$$

4. Complete the proof of Lemma 4, Week 9 notes for the general case $Lu := \sum_{i,j} a_{i,j} \partial_{x_i x_j} u + \sum_i b_i \partial_{x_i} u + cu$.