# Week 1 notes: Math 6451

## 1 PDE: Order, Linear & Nonlinear

A differential equation involving more than one independent variable is called a *partial differential equation*, abbreviated as PDE. The order of the highest derivative occurring in the PDE is defined as the *order* of the PDE. For instance, the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -g(x,y) \tag{1}$$

is a second order PDE for $u$, called *Poisson's* equation in two variables $(x,y)$ with a given *source*[1] $g$. When $g = 0$, (1) is called *Laplace* equation in two variables. Another common PDE that arises in many application is the *heat* or *diffusion* equation. In one space variable in the presence of a *source*, this may be written as

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + g(x,t)\ , \tag{2}$$

where $\kappa$ is a constant called *diffusivity*, and $g$ is called *source*, which is considered known.

As with ODEs, if the differential operator $\mathcal{L}$ is linear, *i.e.*

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L} u_1 + c_2 \mathcal{L} u_2 \tag{3}$$

for constant $c_1, c_2$, then $\mathcal{L}u = g$ is called a linear differential differential equation. Each of (1) and (2) above constitute a linear PDE since we can check easily the linearity of the corresponding differential operators $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, acting on appropriate class of functions[2] Note, that in (2), $g = 0$, then the PDE is linear and homogeneous.

Partial differential equations that are not linear are called *nonlinear*; the following *semi-linear* heat equation is an example:

$$u_t - u_{xx} = u^2, \tag{4}$$

where for notational brevity, the partial derivatives are denoted by subscripts, and it is not unusual to suppress the dependence of $u$ on independent variables, when these are clearly understood.

As for ODEs, linear PDEs are usually simpler to solve or analyze than nonlinear PDEs. Nonlinear PDEs arise in many applications; however, a general theory is much more limited, with explicit solutions few and far between.

## 2 Some physical applications where PDEs arise

PDEs abound in physical sciences and engineering. Here, we give some examples.

[1] In the electro-static context when $u$ is the electro-static potential, $g$ is the charge density

[2] Classically, this would be $\mathbf{C}^2(\mathbb{R}^2)$ in the first case and $\mathbf{C}^2$ in $x$ and $\mathbf{C}^1$ in time for the second. However, this set can be extended further by introducing the concept of weak solutions.

## 2.1 Dispersion of pollutants in a reservoir

Consider predicting concentration $\rho$ (in some units, say $Kg/m^3$) of some pollutant as a function of time $t \geq 0$ for $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ (This domain could be a reservoir for instance). Let the fluid velocity field in $\Omega$ at time $t$ be given by $\mathbf{u}(\mathbf{x}, t)$ in some units (say $m/sec$). Assume the pollutant is *passive*, implying that it does not affect fluid motion. Consider a small but fixed volume $V \subset \Omega$ with smooth boundary $\partial V$ centered at $\mathbf{x}$ as Figure 1. The mass of fluid inside $V$ changes

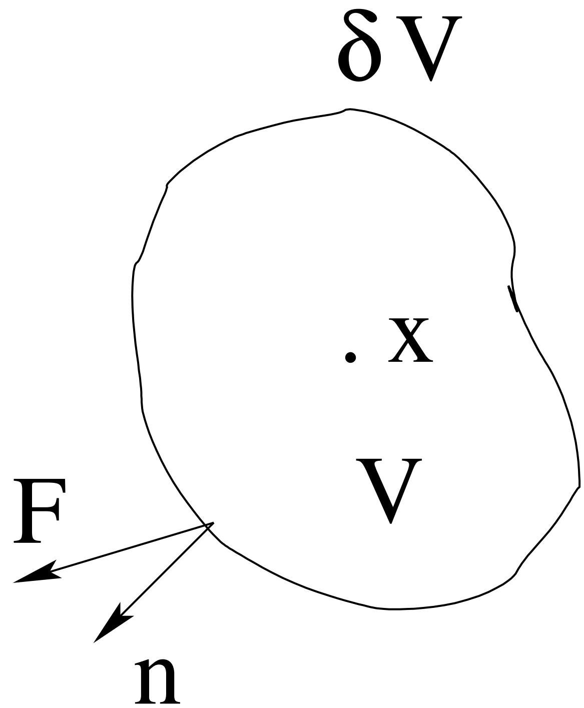


Figure 1: Control Volume V for pollutant

at a rate $\frac{d}{dt} \int_V \rho(\mathbf{y}, t) dV$. If no pollutant, is created or destroyed within $V$, this must equal the rate at which pollutant enters $V$ through its boundary $\partial V$. If $\mathbf{F}(\mathbf{y}, t)$ (measured in a unit like $Kgm/m^2/sec$) is the *flux* across the boundary at a point $\mathbf{y} \in \partial V$ at time $t$, then it is clear that the total inward flux of pollutant must be

$$-\int_{\partial V} \mathbf{F} \cdot \mathbf{n} dA \ ,$$

where $\mathbf{n}$ is the outward normal at $\mathbf{y} \in \partial V$. A physically reasonable expression for flux is:

$$\mathbf{F} = \mathbf{u}\rho - \kappa \nabla \rho$$

where the first term is caused by fluid motion $\mathbf{u}$ carrying pollutants due to what is called *advection* , while the second term is flux due to molecular diffusion, called *Fix's law.* Therefore, mass conservation of pollutants in the volume $V$ leads to

$$\frac{d}{dt}\int_V \rho(\mathbf{y},t)dV = -\int_{\partial V} \mathbf{n}\cdot(\mathbf{u}\rho - \kappa\nabla\rho)\, dA \tag{5}$$

This is the case with no source. If there is a source emitting pollutants at a rate $g(\mathbf{x},t)$ per unit volume (measured in, say, $Kgm/m^3/sec$ units) within $V$, then (5) is replaced by

$$\frac{d}{dt}\int_V \rho dV = -\int_{\partial V} \mathbf{n}\cdot(\mathbf{u}\rho - \kappa\nabla\rho)\, dA + \int_V g dV \tag{6}$$

If we assume $\rho(\mathbf{x},t)$ is $\mathbf{C}^1$ in time $t$, and $\mathbf{C}^2$ in space $\mathbf{x}$, and $g$ to be $\mathbf{C}^0$ in $\mathbf{x}$, then it follows from taking $t$-derivative inside the volume integral and using Gauss's divergence theorem that

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot[(\mathbf{u}\rho - \kappa\nabla\rho] - g\right)(\mathbf{y},t)dV = 0$$

Since this is true for any control volume $V$, it follows that for any $\mathbf{x}\in\Omega$ for $t\geq 0$,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot[(\mathbf{u}\rho - \kappa\nabla\rho] = g \tag{7}$$

However, if we think physically, we cannot expect to predict pollutant density $\rho$ for given $\mathbf{u}$ and $g$ just based on (7) since initial value $\rho(\mathbf{x},0)$ must be relevant as must the conditions at the boundary $\partial\Omega$. So, for prediction of $\rho$, we must append to (7) the initial condition

$$\rho(\mathbf{x},0) = \rho_0(\mathbf{x}) \tag{8}$$

for given $\rho_0$. Appropriate boundary conditions depend very much on the physical circumstances. If we assume no pollutant escapes the reservoir $\Omega$, then the normal flux component at the boundary

$$(\mathbf{u}\rho - \kappa\nabla\rho)\cdot\mathbf{n} = 0 \quad \text{for } \mathbf{x}\in\partial\Omega \tag{9}$$

Further simplifications of the PDE (7) is possible with additional assumptions. If the fluid is incompressible, *i.e.* $\nabla\cdot\mathbf{u} = 0$ and $\kappa$ is a constant, (7) reduces to

$$\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = \kappa\Delta\rho + q \tag{10}$$

where $\Delta = \nabla\cdot(\nabla) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ for $\mathbf{x} = (x_1, x_2, x_3)\in\mathbb{R}^3$ is referred to as the *Laplacian* operator. Equation (10) is referred to as the diffusion-advection equation with source term. In the special case, when there is no source and diffusion is small and we have a one-dimensional constant flow $\mathbf{u}(\mathbf{x},t) = c\,\mathbf{e}_1$, then (10) reduces to

$$\frac{\partial\rho}{\partial t} + c\frac{\partial\rho}{\partial x_1} = 0 \tag{11}$$

whose solution (we can check) is given by

$$\rho(x_1, x_2, x_3, t) = f(x_1 - ct, x_2, x_3) \tag{12}$$

for some arbitrary $\mathbf{C}^1$ function $f$. If we require satisfaction of initial condition (8), then it is clear that $f = \rho_0$. This is a wave travelling along the positive $x_1$ axis with constant speed $c$. This means that in the absence of diffusion, whatever the concentration is at point $(x_1, x_2, x_3)$ at $t = 0$ is now transported downstream to the point $(x_1 + ct, x_2, x_3)$, as expected physically. Note that the approximation where diffusion is totally neglected resulting in solution (12) is generally inconsistent with flux boundary boundary condition (9), since there is no freedom left once we require $f = \rho_0$.

Another interesting limit of (10) is when advection is negligible compared to diffusion, *i.e.* when we set fluid velocity $\mathbf{u} = 0$. We obtain the *diffusion* or *heat* equation[3]

$$\rho_t = \Delta\rho + g \tag{13}$$

The equilibrium solution in this case is found by seeking solution to Poisson equation:

$$\Delta\rho = -g \tag{14}$$

For constant $\mathbf{u}$ and $g$, we can find explicit expression for the full equation (10) in simple geometries, as we will learn later. For more complicated geometries, there exists no explicit method for finding solutions; however, using mathematical analysis, we can prove existence, uniqueness and some valuable properties of such solutions.

It is to be noted that in the derivation of (7) from (6), we assumed *a priori* that solution was indeed smooth enough in $\mathbf{x}$ and $t$ to allow derivatives inside the integral and use divergence theorem. When these conditions are not met, a more fundamental equation based on the physics is given by (6). Indeed, in a general mathematical study of equations such as (7), the assumption of differentiability in $\mathbf{x}$ and $t$ is weakened by introducing the notion of *weak solution.* Instead of satisfying (7), we may require for instance that that $\rho(\mathbf{x}, t)$ satisfy

$$\int_0^\infty \int_\Omega \left[\phi_t\rho + \nabla\phi\cdot(\mathbf{u}\rho + \kappa\nabla\rho) + g\phi\right] dV\ dt = 0 \tag{15}$$

for any smooth function $\phi$ with *compact support*, *i.e.* vanishes outside a bounded set in $\Omega\times\mathbb{R}^+$. If we require that (15) is satisfied and it turns out $\rho \in \mathbf{C}^2(\Omega)\cap\mathbf{C}^1(\mathbb{R}^+)$, then clearly on integration by parts (7) is satisfied. However, the weak solutions satisfiying (15) is more general since they do *not* require solutions to be as differentiable. We will discuss weak solutions in more details later in the quarter.

## 2.2 Vibrating String

Consider a flexible, elastic homogeneous string of length $l$ with constant linear density $\rho$, measured in units of mass/length. Suppose it undergoes transverse vibration like a guitar or violin string. At a given instant of time $t$, the shape locally looks like what is shown in thick outline in Figure 2. Assume that the string motion is restricted to a plane. Let $u(x, t)$ be the vertical displacement from equilibrium at time $t$ at position $x$. For a perfectly flexible string, the tension (force) is directed tangentially along the string. Let $T(x, t)$ denote the magnitude of the tension tension vector $\mathbf{T}(x, t)$. We shall apply Newton's law of motion for the part of the string between

[3] So named because temperature also satisfies the same equaton when heat flux occurs through molecular diffusion only

$x = x_0$ and $x = x_1$. The slope of the string at $x_1$ is clearly $u_x(x_1, t)$. The component of forces in the horizontal direction must be in balance when there is no motion in that direction implying

$$\frac{T(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} = 0 \tag{16}$$

The upward component of the net forces must equal mass times acceleration of the portion of string between $x_0$ and $x_1$. Hence, using arclength $s$ increasing with $x$,

$$\frac{Tu_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - \frac{Tu_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} = \int_{x_0}^{x_1} u_{tt}\rho \frac{ds}{dx} dx \tag{17}$$

In the case, when $u(x,t)$ is $\mathbf{C}^2$ in $x$, we may write (17) as

$$\int_{x_0}^{x_1} \left\{ \left( \frac{Tu_x}{\sqrt{1+u_x^2}} \right)_x - \rho\sqrt{1+u_x^2} u_{tt} \right\} dx$$

Since this is true for any interval $(x_0, x_1)$, it follows that

$$\left( \frac{Tu_x}{\sqrt{1+u_x^2}} \right)_x - \rho\sqrt{1+u_x^2} u_{tt} = 0 \tag{18}$$

On the otherhand (16) implies

$$\partial_x \left( \frac{T}{\sqrt{1+u_x^2}} \right) = 0 \tag{19}$$

In the case when $u_x$ is small, *i.e.* small slope, (18), neglecting quadratic terms in $u_x$, we obtain from (19) that $T$ is a constant, while (18) results in $(Tu_x)_x = \rho u_{tt}$, or

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}} \tag{20}$$

General solution to (20) is of the form

$$u(x,t) = f(x-ct) + g(x+ct) \tag{21}$$

where $f$ and $g$ are arbitrary $\mathbf{C}^2$ functions of $x$ and $t$. This corresponds to superposition of a wave moving to the right and a wave moving to the left. $f$ and $g$ that are determined completely by *initial* and *boundary* conditions. If the string corresponds to $x \in (0, l)$, then fixed end points would correspond to $u(0,t) = u(l,t) = 0$. From physical considerations, initial conditions would correspond to specifying both the initial displacement $u(x, 0)$ and initial velocity $u_t(x, 0)$. We will discuss more in details later on how to find $f$ and $g$ from initial and boundary conditions.

Variations of (20) including friction, transverse elastic force or externally applied forces. Also, it is possible to generalize by introducing concept of appropriate *weak* solution so that (22) would indeed be the weak solution even when $f$ and $g$ are not in $\mathbf{C}^2$.

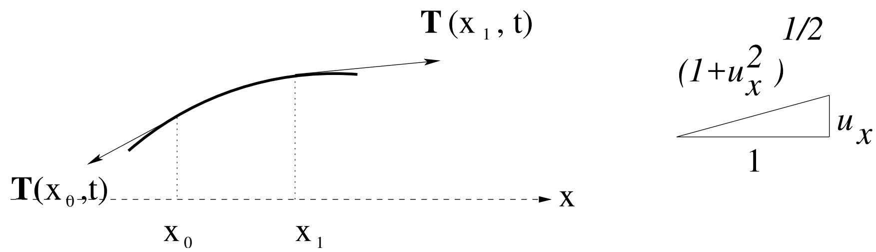


Figure 2: Section of Vibrating string between $x = x_0$ and $x = x_1$

### 2.3 Electro-magnetic waves

Consider Maxwell's equations in free medium for electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. In the Gaussian CGS system of units, they are given by:

$$\nabla \cdot \mathbf{E} = 0 \tag{22}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\mathbf{E}_t = 0 \tag{23}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\mathbf{B}_t = 0 \tag{24}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{25}$$

Taking curl of (24), using (22), (23) and the vector identity $\nabla \times (\nabla \times \mathbf{E}) = -\Delta\mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$, it follows that

$$\Delta\mathbf{E} = \frac{1}{c^2}\mathbf{E}_{tt} \tag{26}$$

In a similar manner, eliminating $\mathbf{E}$ between (23) and (24) and using (25), we obtain

$$\Delta\mathbf{B} = \frac{1}{c^2}\mathbf{B}_{tt} \tag{27}$$

Therefore, each scalar component of $\mathbf{E}$ and $\mathbf{B}$ satisfy the linear wave equation in 2-D or 3-D:

$$\Delta u = \frac{1}{c^2}u_{tt} \tag{28}$$

The solution to (26) and (27) have to be subject again to initial and boundary conditions. A physically reasonable initial condition will be to specify both $\mathbf{E}(\mathbf{x}, 0)$ and $\mathbf{B}(\mathbf{x}, 0)$. For a finite domain $\Omega$ in $\mathbf{x}$, we may specify for instance $\mathbf{E}$ or $\mathbf{B}$ on the boundary $\partial\Omega$ or a combination of components of $\mathbf{E}$ and $\mathbf{B}$ at the boundaries, the latter is common for instance with a conducting boundary.

As in 1-D, the higher dimensional wave equation allows for propagating waves, in this case propagating with speed $c$. If the electro-magnetic wave is emitted due oscillation of current in a loop, then there is a current term on the right of (23). This results in a forcing term on the right of (26) and (27) and we obtain an inhomogeneous wave equation. You can check out texts in electrodynamics (for instance Jackson) for other interesting examples.

### 2.4 Dirichlet, Neumann and Robin Boundary Conditions for 2nd order PDEs

In a more general context of solving wave equation (28), or for that matter any problem with second order spatial derivative like diffusion equation (13) or Laplace equation (14), if we specify $u$ on $\partial\Omega$, this constitutes the *Dirichlet* Boundary conditions. If the normal derivative $\frac{\partial u}{\partial n}$ is specified instead, it is referred to as the *Neumann* BC. A boundary condition in the form of specified $\frac{\partial u}{\partial n} + au = c$ on $\partial\Omega$ will be called a *Robin* boundary condition.

## 3 Well-Posed Problems

A PDE in a domain $\Omega$, together with initial and boundary conditions, is well-posed if the following properties are valid:

1. *Existence*: There is at least one solution to the problem that satisfies all conditions.
2. *Uniqueness*: There is at most one solution.
3. *Stability*: The unique solution depends continuously, with respect to some norm, on initial and boundary conditions. More precisely, let solutions $u_1$ and $u_2$ correspond to initial/boundary data $u_{1,0}$ and $u_{2,0}$. Stability implies that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|u_{1,0} - u_{2,0}\| < \delta$, then $\|u_1 - u_2\| < \epsilon$.

For a physical problem modeled by a PDE, one has to formulate physically realistic auxiliary conditions (like initial and boundary conditions) which together makes a well-posed problem. The mathematician has to prove if a problem is well-posed or not, since modeling usually involves approximation and it is *a priori* unclear whether the approximations are all consistent. Reliance on physical intuition alone is not always enough. If too few auxiliary conditions are given, the problem may not have unique solution and is therefore *underdetermined*. If too many are specified, the solution may not have any soluton and the problem is *overdetermined*.

The stability property (iii) is required in models of physical problems, since data is never measured exactly. You cannot distinguish between a set of data from a tiny perturbation of it (in sense of any physically relevant norm). The solution ought not to be signficantly affected by such tiny perturbations, as otherwise one loses all predictability. A model is not physically sensible without this stability.

For example, if we consider inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t) \quad \text{for } (x,t) \in (0,L) \times (0,\infty) \tag{29}$$

with auxiliary conditions

$$u(x,0) = \phi(x) \quad , \quad u_t(x,0) = \psi(x) \quad ; \quad u(0,t) = g(t) \quad ; \quad u(L,t) = h(t) \tag{30}$$

The data for this problem involves five functions $f$, $g$, $\phi$, $\psi$ and $h$. So, in order for problem to be well-posed, (29) with auxiliary conditions (30) must have a unique solution and the solution has to depend continuously with respect to small changes in each of these five functions.

We can be more precise in our description of continuous dependence on initial/boundary condition. Suppose $u$, $v$ denote solutions to PDEs with with boundary and initial conditions

$u_B, v_B$ and $u_0, v_0$ respectively. Continuous dependence implies that for any any $\epsilon > 0$, there exists $\delta$ so that if $\|u_B - v_B\| + \|u_0 - v_0\| < \delta$ implies $\|u - v\| < \epsilon$. For time evolution problems involving $t \in (0, \infty)$, in defining continous depedence on data, we typically require the above conditions only on $t \in (0, T)$ for fixed $T$.

Consider an example of an ill-posed problem:

$$u_{tt} + u_{xx} = 0 \ \text{ for } \ (x,t) \in \mathcal{D} = (-\infty, \infty) \times (0, \infty) \tag{31}$$

with initial condition

$$u(x,0) = g(x) \ , \ \ u_t(x,0) = h(x) \text{ where } g, h \text{ are } 2\pi \ \text{ periodic function} \tag{32}$$

Note $u = v = 0$ is an obvious solution when $g = h = 0$. We also note on substitution into (31) that $u(x,t) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \ \sinh(nt)$ is a solution to (31) and this satisfies

$$u(x,0) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \ \ u_t(x,0) = e^{-\sqrt{n}} \cos(nx) \tag{33}$$

Now for given $T > 0$ over the interval $(0, T)$ for a finite $\epsilon \neq 0$ (say $\epsilon = 1$), for any $\delta > 0$ we can choose $n \in \mathbb{Z}^+$ large enough so that

$$\|u(\cdot, 0) - v(\cdot, 0)\|_\infty = \frac{1}{n} e^{-\sqrt{n}} < \delta \quad , \quad \|u_t(\cdot, 0) - v_t(\cdot, 0)\|_\infty = e^{-\sqrt{n}} < \delta \ , \tag{34}$$

yet

$$\|u - v\|_\infty = \frac{1}{n} e^{-\sqrt{n}} \sinh(nT) > 1 \tag{35}$$

Therefore, the (31)-(32) does not have continuous dependence on initial condition, at least in the $\| \cdot \|_\infty$ norm, and therefore the problem is *ill-posed*. In general, it is possible for a problem to be ill-posed in one norm and well-posed in another. However, in this particular example, the problem is ill-posed on any physically sensible norm.