Week 10 Lectures, Math 6451, Tanveer

1 An existence proof for Dirichlet problem for $\Delta u = 0$

The purpose of this Week's lectures is to prove existence of solution to the Dirichlet problem for Laplace's equation for a general bounded domain $\Omega \subset \mathbb{R}^n$.

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n \text{ , } u = g \text{ on } \partial\Omega \tag{1}$$

Remark 1 *(Restriction on Domain)* Besides boundedness, it will be assumed that that for each boundary point $\boldsymbol{\xi} \in \partial\Omega$, it is possible to construct ball $B = B_R(\mathbf{y})$ in the exterior of Ω so that $\bar{B} \cap \bar{\Omega} = \{\boldsymbol{\xi}\}$. This is certainly the case if $\partial\Omega$ is a C^2 curve, but is true more generally.

First we recall the following result for a ball:

Theorem 1 *Let $B \subset \mathbb{R}^n$ be a ball of radius R centered at the origin, with $u = g$ on ∂B. Then, for $\mathbf{x} \in B$,*

$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x}|^2}{n\omega_n R} \int_{\partial B} \frac{g(|\mathbf{y}|d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^n} \text{ ,} \tag{2}$$

where ω_n is the volume of an unit Ball in $\mathbb{R}^n$.

PROOF. Recall the Greens function for the Dirichlet problem on a sphere of radius R in $\mathbb{R}^n$ is given by $G(\mathbf{x}, \mathbf{y}) = G_0(|\mathbf{x} - \mathbf{y}|) - G_0\left(\frac{|\mathbf{y}|}{R}|\mathbf{x} - \mathbf{y}^*|\right)$ (see Week 8 notes, page 6), where $\mathbf{y}^* = \frac{R^2}{|\mathbf{y}|^2}\mathbf{y}$, $G_0(r) = \frac{1}{(2-n)n\omega_n} r^{2-n}$ is the Free space Green's function (see Week 8 notes, page 1). We then use the Green's identity involving a harmonic u:

$$u(\mathbf{x}) = \int_{\mathbf{y} \in \partial\Omega} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{y}, \mathbf{x}) g(\mathbf{y}) d\mathbf{y} \tag{3}$$

to obtain the desired statement after some calculations that is left as an exercise. □

1.1 Subharmonic Functions

Definition 2 *$u \in C^0(\bar{\Omega})$ is called subharmonic (superharmonic), if for every ball B with $\bar{B} \subset \Omega$ and each function $h \in C^0(\bar{B})$ that is harmonic in B with $u \leq h$ ($u \geq h$) on ∂B, we have $u \leq h$ ($u \geq h$) in B.*

Remark 2 Note that if $\Delta u \geq 0$ ($\Delta u \leq 0$) with $u \geq h$ ($u \leq h$ on $\partial\Omega$), then from maximum principle u is subharmonic (superharmonic). However, the definition above is weaker in the sense that subharmonic (superharmonic) u need not be in $C^2(\Omega)$. However, we will call $\Delta u \geq 0$ ($\Delta u \leq 0$) as the **strongly** subharmonic (superharmonic) functions.

Lemma 3 *If u is subharmonic and v is superharmonic with $v \geq u$ on $\partial\Omega$, then either $v > u$ in Ω or $v = u$ everywhere.*

PROOF. Suppose $u - v$ assumes its maximum M at some point $\mathbf{x}_0 \in \Omega$, where with $M \geq 0$. If $u - v = M$ through out, then since $\Delta(u - v) = 0$, and $u - v \leq 0$ on $\partial\Omega$, maximum principle implies $M = 0$. Suppose this is not the case. Then, we can choose a ball B centered at $\mathbf{x}_0$ so that $u - v < M$ at some point in B. From continuity of $u - v$, there exists some open set with nonzero measure where $u - v < M$. Suppose $\bar{u}$ and $\bar{v}$ are harmonic functions in B so that on ∂B, $\bar{u} = u$ and $\bar{v} = v$. Then, using strong maximum principle for harmonic functions, we have

$$M = u(\mathbf{x}_0) - v(\mathbf{x}_0) \leq \bar{u}(\mathbf{x}_0) - \bar{v}(\mathbf{x}_0) < \sup_{x \in \partial B} (\bar{u} - \bar{v})(\mathbf{x}) = \sup_{x \leq \partial B} (u(\mathbf{x}) - v(\mathbf{x}) \leq M \ , \quad (4)$$

which is a contradiction. Therefore $u - v < 0$ in Ω, unless $u = v$ everwhere. □

Lemma 4 *Let u be subharmonic in Ω and let B be a ball with $\bar{B} \subset \Omega$. Let $\bar{u}$ be the harmonic function in B satisfying $\bar{u} = u$ on ∂B. Then, the function*

$$U(\mathbf{x}) = \begin{cases} \bar{u}(\mathbf{x}) \text{ for } \mathbf{x} \in B \\ u(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega \setminus B \end{cases} \quad (5)$$

is subharmonic as well with $U(\mathbf{x}) \geq u(\mathbf{x})$.

PROOF. Clearly by assumption $\bar{u}$ is harmonic in B with $u = \bar{u}$ on ∂B; therefore since u is subharmonic in B, it follows $u \leq \bar{u} = U$ in B. In $\Omega \setminus B$, by definition, $u = U$. Therefore for every $\mathbf{x} \in \Omega$, $u(\mathbf{x}) \leq U(\mathbf{x})$. Further, for a harmonic h with $u \leq h$, we note $U = u \leq h$ on ∂B, then harmonicity of of $\bar{u}$ implies that $\bar{u} \leq h$ in $\bar{B}$. Therefore, from definition of of U, for any $\mathbf{x} \in \Omega$, $U(\mathbf{x}) \leq h(\mathbf{x})$. and U is therefore subharmonic. □

Definition 5 *For any subharmonic u, U defined by 5 is called the harmonic lifting of u in B.*

Lemma 6 *If $\{u_1, u_2, ..u_m\}$ is a set of subharmonic functions, so is*

$$U = \max \{u_1, u_2, ..u_m\}$$

PROOF. The proof is left as an exercise. □

1.2 Equicontinuity and convergence of family of harmonic functions

We now introduce the concept of equi-continous family of functions. This will be useful later in our proof of existence of the Dirichlet problem.

Definition 7 *A family of functions $\{f_m\}_{m=1}^{\infty}$ defined in $S \subset \mathbb{R}^n$ is equicontinuous at $\mathbf{x} \in S$ if, for every $\epsilon > 0$, there exists δ indpendent of m so that $|\mathbf{y} - \mathbf{x}| < \delta$ implies $|f_m(\mathbf{y}) - f_m(\mathbf{x})| < \epsilon$.*

Remark 3 Note that if the the set S is compact (*i.e.* closed and bounded set in $\mathbb{R}^n$), then $\{f_m\}$ is uniformly equi-continuous, *i.e.* δ in the above definition is independent of $\mathbf{x}$. We also note that equi-continuity of a family $\{f_m\}$ at $\mathbf{x}$ follows if we can show that $f_m'(\mathbf{x})$ exists and is bounded independent of m.

Theorem 8 *(Arzela-Ascoli Theorem) Let $\{f_m\}_{m=1}^{\infty}$ be a sequence of equi-continuous functions on a compact set $S \subset \mathbb{R}^n$ with $|f_m(\mathbf{x})| \leq M$, independent of m. Then, there exists a subsequence $\left\{f_{m_j}\right\}_{j=1}^{\infty}$ which converges uniformly on S*

PROOF. Since S is compact, we take a sequence of points $\{\mathbf{x}_i\}_{i=1}^{\infty}$ dense in S. Consider $\{f_m(\mathbf{x}_1)\}_m$ which is is a bounded sequence of numbers, and therefore has a convergent subsequence. Call this subsequence $m_{1,j}$, *i.e.* $f_{m_{1,j}}(\mathbf{x}_1)$ converges as $j \to \infty$. Since $\left\{f_{m_{1,j}}(\mathbf{x}_2)\right\}$ is also a bounded sequence of real numbers, there exists a subsequence of $m_{1,j}$ call it $m_{2,j}$ so that $f_{m_{2,j}}(\mathbf{x}_2)$ converges. Note that since $m_{2,j}$ is a subsequence of $m_{1,j}$, $f_{m_{2,j}}(\mathbf{x}_1)$ also converges. We keep going by taking subsequence $m_{3,j}$ of $m_{2,j}$ so that $f_{m_{3,j}}(\mathbf{x}_3)$ converges and from construction also converges at $\mathbf{x}_1$ and $\mathbf{x}_2$. It is clear that in this manner the sequence $g_j := f_{m_{j,j}}$ converges at each point $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ .. as $j \to \infty$. Now, we want to prove that g_m is uniformly Cauchy in S. Take any $\epsilon > 0$. Since g_m is uniformly continuous (because it is a subsequence of f_m) there exists $\delta > 0$ independent of m, so that if $\mathbf{x}, \mathbf{y} \in S$ with $\left|\mathbf{x} - \mathbf{y}\right| < \delta$, then $\left|f_m(\mathbf{x}) - f_m(\mathbf{y})\right| < \epsilon/3$. Also, since S is compact, there exists a finite set of points $\{\mathbf{x}_1, \mathbf{x}_2, ..\mathbf{x}_K\}$ so that any $\mathbf{x} \in S$ is in a δ neighborhood of some $\mathbf{x}_j$. We choose N large enough so that for $m, l \geq N$, $\left|g_m(\mathbf{x}_i) - g_l(\mathbf{x}_i)\right| < \frac{\epsilon}{3}$ for any $i = 1, ..K$. Then,

$$\left|g_m(\mathbf{x}) - g_l(\mathbf{x})\right| \leq \left|g_m(\mathbf{x}) - g_m(\mathbf{x}_j)\right| + \left|g_m(\mathbf{x}_j) - g_l(\mathbf{x}_j)\right| + \left|g_l(\mathbf{x}_j) - g_l(\mathbf{x})\right| < \epsilon \quad (6)$$

□

Theorem 9 *Let $\Omega \subset \mathbb{R}^n$. Let f_m be a sequence of harmonic functions on Ω which is uniformly bounded, i.e. $|f_m(\mathbf{x})| \leq M$ for every $\mathbf{x} \in \Omega$. Then, f_m has a subsequence that converges to a harmonic function on Ω, uniformly on compact subsets of Ω.*

Proof. Define compact subsets $\Omega_k \subset \Omega$ so that

$$\Omega_k = \left\{ \mathbf{x} \in \Omega : |\mathbf{x}| \leq k, \operatorname{dist}(\mathbf{x}, \partial\Omega) \geq \frac{1}{k} \right\} \tag{7}$$

It is clear that $\Omega = \cup_{k=1}^{\infty} \Omega_k$. If $\mathbf{x} \in \Omega_k$ For each point $\mathbf{x}_0 \in \Omega_k$, we apply Theorem 1 on a ball $B_{1/(2k)}(\mathbf{x})$ centered at $\mathbf{x}_0$ to obtain

$$\nabla f_m(\mathbf{x}_0) = \frac{1}{n\omega_n} \int_{|\mathbf{y}-\mathbf{x}_0| = \frac{1}{2k}} \left(\nabla_{\mathbf{x}} \frac{1/(4k^2) - |\mathbf{x}-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{y}|^n} \right)_{\mathbf{x}=\mathbf{x}_0} f_m(\mathbf{y}) d\mathbf{y} \ , \tag{8}$$

which immediately implies that

$$\left| \nabla f(\mathbf{x}_0 \right| \leq CM \tag{9}$$

for some constant C independent of m and $\mathbf{x}_0$. From Arzela-Ascoli theorem, there exists a subsequence that converges in Ω_k. Through a standard diagonalization procedure, it is possible to obtain a subsequence of a subsequence of a subsequence that converges for each $\mathbf{x}$ to a funciton f and that this convergence is uniform for each compact set Ω_k. Further, using Theorem 1 again, for any fixed point $\mathbf{x} \in B_{1/(2k)}(\mathbf{x}_0)$

$$\begin{aligned} f(\mathbf{x}) = \lim_{j\to\infty} f_{m,j}(\mathbf{x}) = \frac{(1/(2k)^2 - |\mathbf{x}|^2}{n\omega_n} \int_{|\mathbf{y}-\mathbf{x}_0|=1/(2k)} \lim_{j\to\infty} \frac{f_{m,j}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} d\mathbf{y} \\ = \frac{(1/(2k)^2 - |\mathbf{x}|^2}{n\omega_n} \int_{|\mathbf{y}-\mathbf{x}_0|=1/(2k)} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} d\mathbf{y} \end{aligned} \tag{10}$$

It may be verified directly (by applying operator $\Delta_{\mathbf{x}}$) that the right most term is harmonic function in $\mathbf{x} \in B_{1/(2k)}(\mathbf{x}_0)$ for any *a priori* continuous f; hence from the equation above f is harmonic. By choosing $\mathbf{x}_0$ suitably, the harmonicity of f for any $\mathbf{x} \in \Omega_k$ follows. □

1.3 Proof of solution to $\Delta u = 0$ for Dirichlet B.C.

Definition 10 *A subsolution v for the Dirichlet problem with data $g \in C^0(\partial\Omega)$ is defined to be a function $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $\Delta v \geq 0$ with $v \leq g$ on $\partial\Omega$ Similarly, a super solution w for the Dirichlet problem is defined to be a function $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $\Delta w \leq 0$ with $v \geq g$ on $\partial\Omega$.*

Remark 4 Prior discussions related to use of maximum principle x imply that a subsolution is subharmonic and a supersolution is superharmonic and that $v \leq w$ for any sub-super solution pair (v, w) for given g. Further, the set of sub or super solution is nonempty for any $g \in C^0(\partial\Omega)$ since a sufficiently small constant is a subsolution, where as a suffiently large constant is a supersolution.

Definition 11 *Define S_g to be the set of all subsolutions corresponding to given $g \in C^0(\partial\Omega)$.*

$$u = \sup_{v \in S_g} v \tag{11}$$

Remark 5 The sup in the above definition always exist since any supersolution is an upper bound.

Lemma 12 *u defined above satisfies $\Delta u = 0$ in Ω.*

Proof. For some $\mathbf{x} \in \Omega$ consider a sequence of $v_m \in S_g$ with the property

$$\lim_{m \to \infty} v_m(\mathbf{x}) = u(\mathbf{x}) \tag{12}$$

Note $v_m(\mathbf{x})$ is bounded from above, and v_m can be chosen to be bounded from below (by replacing if necessary v_m by $\max\{v_0, v_m\}$), where v_0 is any subsolution). Choose R so that $B = B_R(\mathbf{x}) \subset \Omega$ and let V_m be the harmonic lifting of v_m (see Definition 5) with respect to B. Then $v_m \leq V_m \to u$ as $m \to \infty$ since V_m is itself subharmonic and cannot exceed the u. So, applying Theorem 8, there exists a subsequence $\left\{V_{m_j}\right\}_{j=1}^{\infty}$ that converges on B, the convergence being uniform in compact subset of B, and the function it converges to, call it v, must be harmonic. Clearly $v \leq u$ in B. We shall now prove that $v = u$ in B, which would imply that u is harmonic in Ω since the argument can be repeated in any ball with different centers in Ω.

Assume this is not the case; that there exists $\mathbf{y} \in B$ so that $v(\mathbf{y}) < u(\mathbf{y})$. Then there exists fuctinion $W \in S_g$ such that $v(\mathbf{y}) < W(\mathbf{y})$. Let $w_j = \max\left(W, V_{m_j}\right)$ and let W_j be the harmonic lifting of w_j with respect to B. As before a subsequence of W_j will converge to a function w which is harmonic in B. Clearly, we have $v \leq w$ on B (from construction of w_j and the harmonic lifting process) and further, $v(\mathbf{x}) = w(\mathbf{x})$, since

$$u(\mathbf{x}) \geq w(\mathbf{x}) \geq v(\mathbf{x}) = \lim_{j \to \infty} V_{m_j}(\mathbf{x}) \geq \lim_{j \to \infty} v_{m_j}(\mathbf{x}) = u(\mathbf{x})$$

By strong maximum principle $v = w$ in B, contradicting the choice of W. Therefore, $v = w$ in B. Since this process can be repeated for ball centered at any $\mathbf{x} \in \Omega$, it follows that $u(\mathbf{x})$ thus constructed is harmonic. □

Remark 6 We now wish to prove that as $x \to \partial\Omega$, with u constructed as before, has the property $u \to g$. Recall from restriction on the domain Ω, we we assumed that for each boundary point $\boldsymbol{\xi} \in \partial\Omega$, it is possible to construct ball $B = B_R(\mathbf{y})$ in the exterior of Ω so that $\bar{B} \cap \bar{\Omega} = \{\boldsymbol{\xi}\}$.

Definition 13 *Define*

$$w(\mathbf{x}) = R^{2-n} - |\mathbf{x} - \mathbf{y}|^{2-n} \text{ , if } n \geq 3$$

$$w(\mathbf{x}) = \log \frac{|\mathbf{x} - \mathbf{y}|}{R}$$

It is easily checked that w is harmonic (and therefore superharmonic) and that $w(\mathbf{x}) > 0$ in Ω.

Lemma 14 *Let u be the harmonic function in Ω constructed above and let $\boldsymbol{\xi} \in \partial\Omega$. Then $u(\mathbf{x}) \to g(\xi)$ as $\mathbf{x} \to \boldsymbol{\xi}$*

Proof. For $\epsilon > 0$, let $M = \sup_{\partial\Omega} |g|$. Let w be as defined above and we first choose δ so that $\Big|g(\mathbf{x}) - g(\boldsymbol{\xi})\Big| < \epsilon$ for $\Big|\mathbf{x} - \boldsymbol{\xi}\Big| < \delta$. Now choose k large enough so that $kw(\mathbf{x}) > 2M$ for $\mathbf{x} \in \Omega$, $\Big|\mathbf{x} - \boldsymbol{\xi}\Big| \geq \delta$. The function $g(\boldsymbol{\xi}) + \epsilon + kw(\mathbf{x})$ and $g(\boldsymbol{\xi}) - \epsilon - kw(\mathbf{x})$ are respectively supersolution and sub solution in Ω corresponding to g and therefore,

$$g(\boldsymbol{\xi}) + \epsilon + kw(\mathbf{x}) \geq u(\mathbf{x}) \geq g(\boldsymbol{\xi}) - \epsilon - kw(\mathbf{x})$$

Since $w(\mathbf{x}) \to 0$ as $\mathbf{x} \to \xi$, the Lemma follows immediately. $\square$