Week 11 Lectures, Math 6451, Tanveer

1 Nonlinear hyperbolic equations

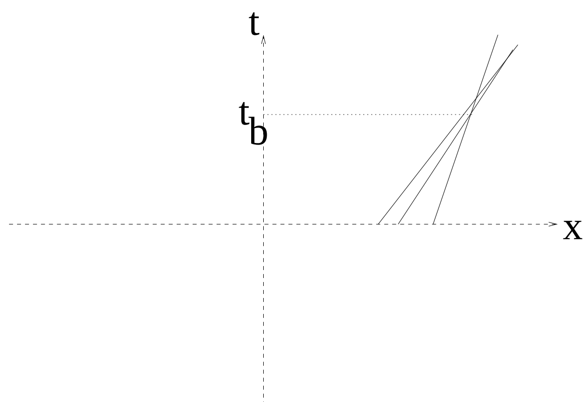

Figure 1: Intersecting characteristics for $t > t_b$

As discussed earlier in this course, in the context of Burger's equation $u_t + uu_x = 0$, solutions to nonlinear hyperperbolic equations are usually characterized by singularity formation. Classical solution do not generally exist beyond a finite time and we are forced to consider weak solutions. However, typically such solutions are not unique and additional conditions are needed to get a unique solution. To see this recall that solution to inviscid Burger's equation

$$u_t + uu_x = 0 \ , u(x,0) = F(x) \ , \ \text{for } x \in \mathbb{R} \ , \ t > 0 \tag{1}$$

is given by

$$u = F(\xi) \ , \ \text{where } x = \xi + tF(\xi) \tag{2}$$

In the above ξ is implicitly determined in terms of (x,t) by inverting the above relation, which is possible for t small enough so that

$$1 + tF'(\xi) > 0 \tag{3}$$

There is no restriction on t when $F' > 0$, and the classical solution exists for all time since the mapping $\xi \rightarrow x$ is 1-1 for any time $t > 0$. However, generically, there exists nonempty set S_- where $F' < 0$ and classical solutions exists only for $t \in (0, t_b)$, where

$$t_b = \inf_{\xi \in S_-} \frac{1}{\left| F'(\xi) \right|} \tag{4}$$

For $t > t_b$, characteristics intersect and we have more than one ξ corresponding to the same x in some region in the $x - t$ plane, as sketched in Fig. 1. Classical solutions do not make

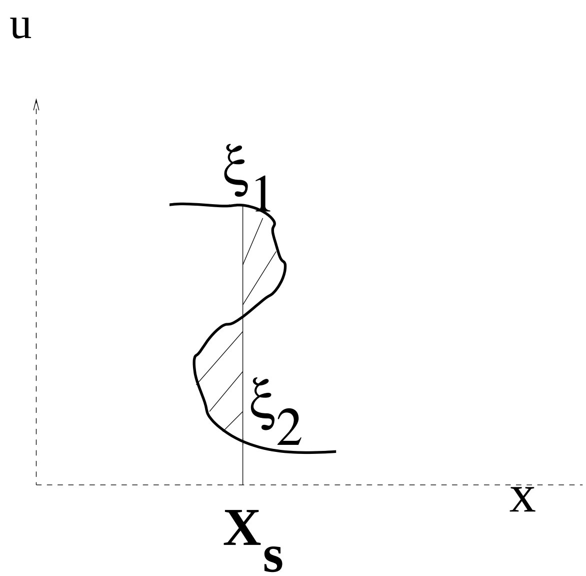

Figure 2: Obtaining weak solution by applying equal area rule for $t > t_b$

sense for $t > t_b$. Reconstruction of $u(x,t) = F(\xi)$ through local inversion of $x = \xi + tF(\xi)$ leads to a multi-valued function sketched in Fig. 2 for some range of x.

We now construct a weak solution, for which u jumps across $x = X_s(t)$ for $t > t_b$. This is shown by the vertical line that divides the inverted S-shaped region in the $u - x$ plane into two shaded regions show in Fig. 2. Let $\xi = \xi_1$ correspond to the uppermost intersection point of $x = X_s(t)$ with the S shaped curve in Fig. 2, while $\xi = \xi_2 > \xi_1$ corresponds to the lowermost intersection point of this vertical line. We note that $\xi \to x$ mapping is 1-1 for $\xi \in (-\infty, \xi_1)$ and for $\xi \in (\xi_2, \infty)$. So,

$$u(x,t) = F(\xi(x,t)) \text{ ,for } \xi \notin (\xi_1, \xi_2) \tag{5}$$

will satisfy $u_t + uu_x = 0$ for $x < X_s(t)$ and $x > X_s(t)$. We note however that $u(X_s^-(t), t) = F(\xi_1)$ and $u(X_s^+(t), t) = F(\xi_2)$ and there is a jump of $F(\xi_2) - F(\xi_1)$ of u across the

$x = X_s(t)$, referred to usually as a *shock*. Since both characteristics corresponding to ξ_1 and ξ_2 correspond to the same $x = X_s$, we obtain

$$x = X_s(t) = \xi_1 + tF(\xi_1) = \xi_2 + tF(\xi_2) \tag{6}$$

To obtain a weak reformulation of $u_t + uu_x = 0$, we write this as $u_t + (u^2/2)_x = 0$, multiply by by an arbitrary test function $\phi(x,t)$ and integrate in x and t. Integration by parts gives

$$0 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\phi_t u + \frac{1}{2} u^2 \phi_x \right) dxdt \tag{7}$$

We examine the conditions on $\dot{X}_s$ so that the solution constructed in (5) with jump across $x = X_s(t)$ for $t > t_b$ satisfies (7), which may be written as

$$0 = \int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s(t)} + \int_{X_s(t)}^{\infty} \right\} \left(\phi_t u + \frac{1}{2} u^2 \phi_x \right) dxdt \tag{8}$$

We note

$$\begin{aligned} &\int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s(t)} + \int_{X_s(t)}^{\infty} \right\} \phi_t u dxdt = \int_{\mathbb{R}} \partial_t \left[\left\{ \int_{-\infty}^{X_s(t)} + \int_{X_s(t)}^{\infty} \right\} \phi u dxdt \right] \\ &- \int_{\mathbb{R}} \left\{ -\dot{X}_s \left[u(X_s^+(t), t) - u(X_s^-(t), t) \right] \phi(X_s(t), t) \right\} dt - \int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s^-(t)} + \int_{X_s^+(t)}^{\infty} \right\} u_t \phi dxdt \end{aligned} \tag{9}$$

while

$$\begin{aligned} \int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s(t)} + \int_{X_s(t)}^{\infty} \right\} \phi_x \frac{u^2}{2} dxdt &= \int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s(t)} + \int_{X_s(t)}^{\infty} \right\} \partial_x \left[\phi \frac{u^2}{2} \right] dxdt \\ &= \int_{\mathbb{R}} \left\{ -\frac{1}{2} \left[u^2(X_s^+(t), t) - u^2(X_s^-(t), t) \right] \phi(X_s(t), t) \right\} dt \\ &\qquad - \int_{\mathbb{R}} \left\{ \int_{-\infty}^{X_s^-(t)} + \int_{X_s^+(t)}^{\infty} \right\} u u_x \phi dxdt \end{aligned} \tag{10}$$

Therefore, adding (9) and (10), and using $u_t + uu_x = 0$, for $(x,t) \neq (X_s(t), t)$, we obtain

$$0 = \int_{t_b}^{\infty} dt \phi(X_s(t), t) \left\{ \dot{X}_s(t) \left[u(X_s^+(t), t) - u(X_s^-(t), t) \right] - \frac{1}{2} \left[u^2(X_s^+(t), t) - u^2(X_s^-(t), t) \right] \right\} \tag{11}$$

Since this has to be true for arbitary test function ϕ, we obtain a weak solution if

$$\dot{X}_s(t) = \frac{1}{2}\left(u(X_s^+(t),t) + u(X_s^-(t),t)\right) = \frac{1}{2}\left[F(\xi_2) + F(\xi_1)\right], \tag{12}$$

From (6), we also get

$$\dot{X}_s(t) = (1 + tF'(\xi_1))\,\dot{\xi}_1 + F(\xi_1) \tag{13}$$

$$\dot{X}_s(t) = (1 + tF'(\xi_2))\,\dot{\xi}_2 + F(\xi_2) \tag{14}$$

Averaging the two equation and using (12), we obtain

$$0 = \frac{1}{2}\dot{\xi}_1\,(1 + tF'(\xi_1)) + \frac{1}{2}\dot{\xi}_2\,(1 + tF'(\xi_2)) \tag{15}$$

Using $t = -\frac{\xi_2 - \xi_1}{F(\xi_2) - F(\xi_1)}$, which follows from (6), we obtain from (15)

$$0 = \frac{\dot{\xi}_1}{2}\,(F(\xi_1) - F(\xi_2)) - \frac{\dot{\xi}_1}{2}\,(\xi_1 - \xi_2)\,F'(\xi_1) + \frac{\dot{\xi}_2}{2}\,(F(\xi_1) - F(\xi_2)) - \frac{\dot{\xi}_2}{2}\,(\xi_1 - \xi_2)\,F'(\xi_2)\,, \tag{16}$$

which leads to

$$\frac{1}{2}\left\{\dot{\xi}_1 F'(\xi_1) + \dot{\xi}_2 F'(\xi_2)\right\}(\xi_1 - \xi_2) + \frac{1}{2}\,(F(\xi_1) + F(\xi_2))\left(\dot{\xi}_1 - \dot{\xi}_2\right) = \dot{\xi}_1 F(\xi_1) + \dot{\xi}_2 F(\xi_2) \tag{17}$$

Integration in time from $t = t_b$ when $\xi_1 = \xi_2$, *i.e.* when shock forms gives rise to

$$\frac{1}{2}\,(F(\xi_1) - F(\xi_2))\,(\xi_1 - \xi_2) = \int_{\xi_1}^{\xi_2} F(\xi)d\xi \tag{18}$$

This corresponds to the equal area rule, requiring that the two shaded region on two sides of the vertical line $x = X_s$ in Fig. 2 have equal areas since

$$0 = \int_{\xi_1}^{\xi_2} u(\xi)x_\xi d\xi = \int_{\xi_1}^{\xi_2} F(\xi)\,(1 + tF'(\xi))\,d\xi = \int_{\xi_1}^{\xi_2} F(\xi)d\xi + \frac{t}{2}\left(F^2(\xi_2) - F^2(\xi_1)\right)\ , \tag{19}$$

which on using $t = -\frac{\xi_2 - \xi_1}{F(\xi_2) - F(\xi_1)}$ leads to (18).

However, mathematically, equal area rule (18) (or the equivalent expression (12) for shock speed $\dot{X}_s$) is *not* the only possibility for weak solution. Instead of putting inviscid

Burger's equation in the *i.e.* conservation form $u_t + (u^2/2)_x = 0$ and multiplying by ϕ to obtain a weak formulation, one can instead note that Burger's equation is equivalent to

$$\partial_t \left(\frac{u^2}{2}\right) + \partial_x \left(\frac{u^3}{3}\right) = 0 \tag{20}$$

To obtain an alternate weak formulation, we multiply (20) by test function ϕ and integrate by parts to obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{1}{2}u^2\phi_t + \frac{1}{3}u^3\phi_x\right) dxdt = 0 \tag{21}$$

Going through the same arguments as before, we obtain a weak solution with $x = X_s(t)$ determined by

$$\frac{1}{2}\dot{X}_s \left(F^2(\xi_2) - F^2(\xi_1)\right) = \frac{1}{3}\left(F^3(\xi_2) - F^3(\xi_1)\right) \tag{22}$$

The above gives rise to a distinctly motion of the shock motion than (12); therefore, weak solutions are not unique.

To determine which weak solution is appropriate to the physical problem we can insert physically appropriate dissipation in the problem. Suppose, we determine νu_{xx} is an appropriate viscous correction to right hand side of Burger's equation. We now examine the limiting solution as $\nu \to 0^+$.

1.1 Viscous Burger's Equation and Cole-Hopf transformation

Consider the viscous Burger's equation

$$u_t + uu_x = \nu u_{xx} \ , x \in \mathbb{R} \ \ , \ t > 0 \ , \text{with } u(x,0) = F(x) \tag{23}$$

We seek solution in the form:

$$u = -\frac{2\nu\psi_x}{\psi} \tag{24}$$

After some algebra, we you may check that ψ satisfies the heat equation

$$\psi_t = \nu\psi_{xx} \ , \text{with } \psi(x,0) = \psi_0(x) := \exp\left[-\frac{1}{2\nu}\int_0^x F(t)dt\right] \tag{25}$$

The transformation (24) which converts the nonlinear equation (23) into the linear heat equation (25) is usually referred to as the **Cole-Hopf** transformation. Recall that the solution to (25) is given by

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \psi_0(y) \exp\left[-\frac{(x-y)^2}{2\nu t}\right] \tag{26}$$

Therefore, using (24)

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t} \exp\left[-\frac{G(y;x,t)}{2\nu}\right] dy}{\int_{-\infty}^{\infty} \exp\left[-\frac{G(y;x,t)}{2\nu}\right] dy} \tag{27}$$

where

$$G(y;x,t) = \frac{(y-x)^2}{2t} + \int_0^y F(y')dy' \tag{28}$$

The following Lemma will be useful in determining the asymptotic behavior of (27) as $\nu \to 0^+$.

Lemma 1 *Assume $\int_a^b \left|Q(y)\right| e^{-\nu^{-1}P(y)} dy$ exists for $\nu \in (0, \nu_0)$ for some $\nu_0 > 0$ and $P \in C^3(a,b)$, with $P' = 0$ at $y = y_0 \in (a,b)$, while $P'(y) \neq 0$ for $y \neq y_0$ in (a,b). Also assume $P''(y_0) > 0$ and Q is continuous in a neighborhood of y_0. Then, as $\nu \to 0^+$,*

$$\int_a^b Q(y)e^{-\nu^{-1}P(y)}dy = \sqrt{\frac{2\pi\nu}{P''(y_0)}} Q(y_0)e^{-\nu^{-1}P(y_0)} \left(1 + o(1)\right) \tag{29}$$

Proof. From given condition P has a minimum at y_0. Choose $\epsilon > 0$. Choose $\delta = \nu^{5/12}$ and $\nu > 0$ small enough so that for $y \in (y_0 - \delta, y_0 + \delta)$,

$$\left|Q(y) - Q(y_0)\right| \leq \epsilon \tag{30}$$

$$\int_{(a,b)\setminus(y_0-\delta, y_0+\delta)} \left|Q(y)\right| \exp\left[-\nu^{-1}P(y)\right] dy \leq \epsilon \nu^{1/2} e^{-P(y_0)} \tag{31}$$

$$\exp\left[-\nu^{-1}\left(-P(y) + P(y_0) + \frac{1}{2}P'(y_0)(y-y_0)^2\right)\right] < \epsilon \text{ , for } y \in (y_0 - \delta, y_0 + \delta) \tag{32}$$

The conditions (30)-(32) are possible to satisfy for small enough ν since $Q \in C^0$ in a neighborhood of y_0, $P''(y_0) > 0$ with lower bound independent of ν and $P \in C^3(a,b)$. In particular as far as showing (31), we note that for some $C > 0$ independent of ν,

$$\nu^{-1}\left|P\left(y_0 \pm \nu^{5/12}\right) - P(y_0)\right| \geq C\nu^{-1/6} \tag{33}$$

We break up the $\int_a^b = \int_a^{y_0-\delta} + \int_{y_0+\delta}$. We note that

$$e^{\nu^{-1}P(y_0)} \int_{y_0-\delta}^{y_0+\delta} Q(y) \exp\left[-\nu^{-1}P(y)\right] dy = Q(y_0) \int_{y_0-\delta}^{y_0+\delta} \exp\left[-\nu^{-1}\left(P(y) - P(y_0)\right)\right] dy$$
$$+ \left\{ \int_{y_0-\delta}^{y_0+\delta} \left(Q(y) - Q(y_0)\right) \exp\left[-\nu^{-1}\left(-P(y) + P(y_0)\right)\right] dy \right\} \tag{34}$$

From conditions (30)-(32), it follows that

$$\int_{y_0-\delta}^{y_0+\delta} Q(y) \exp\left[-\nu^{-1}P(y)\right] dy \sim \sqrt{\frac{2\pi\nu}{P''(y_0)}} e^{-\nu^{-1}P(y_0)} (1 + C\epsilon) \tag{35}$$

Therefore, the lemma follows. □

Corollary 2 *In the case $P(y)$ has multiple minimum in the interval (a, b) at points y_j, for $j = 1, ..n$ with $P''(y_j) > 0$, while other conditions of previous Lemma hold, then*

$$\int_a^b Q(y) e^{-\nu^{-1}P(y)} dy = \sum_{j=1}^{n} \sqrt{\frac{2\pi\nu}{P''(y_j)}} Q(y_j) e^{-\nu^{-1}P(y_j)} (1 + o(1)) \tag{36}$$

Proof. We simply further subdivide the interval (a, b) into smaller intervals so that there exists only one minimum in each sub-interval. Applying previous Lemma, the corollary follows. □

Remark 1 *When multiple minimum are present, but one of the $P(y_j)$ is smaller than the rest, then only one contribution need to retained in (36) since others are exponentially small in ν.*

Using the above Lemma and Corollary, it follows that as $\nu \to 0^+$, we have from (27)

$$\int_{-\infty}^{\infty} \frac{x - y}{t} e^{-\frac{G(y;x;t)}{2\nu}} dy \sim \frac{(x - \xi)}{t} \sqrt{\frac{4\pi\nu}{G''(\xi)}} \exp\left[-\frac{G(\xi; x, t)}{2\nu}\right] , \tag{37}$$

$$\int_{-\infty}^{\infty} \frac{x - y}{t} e^{-\frac{G(y;x;t)}{2\nu}} dy \sim \sqrt{\frac{4\pi\nu}{G''(\xi)}} \exp\left[-\frac{G(\xi; x, t)}{2\nu}\right] , \tag{38}$$

where $y = \xi$ is determined from $G_y(y; x, t) = 0$, *i.e.*

$$F(\xi) - \frac{(x - \xi)}{t} = 0 \text{ , implying } x = \xi + tF(\xi) \tag{39}$$

and from (37) and (38) used in (27), we obtain as $\nu \to 0^+$,

$$u(x, t) \sim \left(\frac{\xi - x}{t}\right) = F(\xi) \tag{40}$$

Also, note that $G_{yy}(\xi; x, t) = \frac{1}{t} + F'(\xi)$. Therefore, as $\nu \to 0$, we recover the solution to $u_t + uu_x = 0$ obtained through method of characteristic. However, when there are two local minimum points of G, *i.e* $G_y(y; x, t) = 0$ for $y = \xi_1, \xi_2$, this corresponds to intersection of multiple characteristics and we know from prior discussion that classical solutions of $u_t + uu_x = 0$ will not make sense. Let's examine the limit $\nu \to 0^+$ of the viscous Burger's equation. Assume ξ_1, ξ_2, are distinct roots of $G_y(y; x, t) = 0$ for given (x, t). Then,

$$\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{G(y;x;t)}{2\nu}} dy \sim \frac{(x-\xi_1)}{t}\sqrt{\frac{4\pi\nu}{G''(\xi_1)}} \exp\left[-\frac{G(\xi_1)}{2\nu}\right] + \frac{(x-\xi_2)}{t}\sqrt{\frac{4\pi\nu}{G''(\xi_2)}} \exp\left[-\frac{G(\xi_2)}{2\nu}\right] \tag{41}$$

$$\int_{-\infty}^{\infty} e^{-\frac{G(y;x;t)}{2\nu}} dy \sim \sqrt{\frac{4\pi\nu}{G''(\xi_1)}} \exp\left[-\frac{G(\xi_1; x, t)}{2\nu}\right] + \sqrt{\frac{4\pi\nu}{G''(\xi_2)}} \exp\left[-\frac{G(\xi_2; x, t)}{2\nu}\right] \tag{42}$$

If $G(\xi_1) < G(\xi_2)$, the smallness of ν implies that only the terms containing the exponential $\exp\left[-G(\xi_1; x, t)\right]$ is important, the other one being exponentially smaller. Therefore, we will have in such cases

$$u(x, t) \sim\sim \left(\frac{x-\xi_1}{t}\right) = F(\xi_1) \ , \tag{43}$$

while for $G(\xi_1) > G(\xi_2)$, we have

$$u(x, t) \sim \frac{x-\xi_2}{t} = F(\xi_2) \ , \tag{44}$$

On the otherhand, if $G(\xi_1; x, t) = G(\xi_2; x, t)$, then both exponentials in (41) and (42) are important. Note that this happens when

$$\int_0^{\xi_1} F(y)dy + \frac{(x-\xi_1)^2}{2t} = \int_0^{\xi_2} F(y)dy + \frac{(x-\xi_2)^2}{2t} \tag{45}$$

Further since $G_y(\xi_1; x, t) = 0 = G_y(\xi_2; x, t)$, it follows that

$$x = \xi_1 + tF(\xi_1) = \xi_2 + tF(\xi_2) \tag{46}$$

Using this, (45) may be rewritten as

$$\frac{1}{2}\left(F(\xi_1) + F(\xi_2)\right)(\xi_1 - \xi_2) = \int_{\xi_1}^{\xi_2} F(\eta')d\eta' \tag{47}$$

This is the equal area rule obtained for weak solutions discussed before. Only one of the weak solution formulation is consistent with the limit $\nu \to 0^+$.

2 Shock Structure

We now seek to determine behavior of u close to $x = X_s(t)$ for small ν. This can be determined by looking for traveling wave solution $u = u(x - ct)$ to the viscous Burger's equation. We obtain

$$-cu_X + uu_X = \nu u_{XX} \tag{48}$$

We seek solution to (48) for which $u \to u_1$ as $X \to -\infty$ and $u \to u_2$ as $X \to \infty$, where $u_1 > u_2$. The velocity of the shock $c = \frac{1}{2}(u_1 + u_2)$. We note on integration

$$-cu + \frac{u^2}{2} = \nu u_X - \frac{A}{2} \text{ , where } A = 2cu_1 - u_1^2 = u_1 u_2 \tag{49}$$

Then, separation of variable leads to

$$\frac{X}{2\nu} = \int^u \frac{du'}{(u' - u_1)(u' - u_2)} + B = \frac{1}{u_1 - u_2}\left(\log \frac{u_1 - u}{u - u_2}\right) \tag{50}$$

Therefore,

$$\frac{u_1 - u}{u - u_2} = B \exp\left[\frac{(u_1 - u_2)}{2\nu} X\right] \tag{51}$$

We may set $B = 1$ with appropriate choice of origin of X. We may note that this solution is expected to describe the inner-structure of any shock, even time dependent ones, since the above calculation only relies on the variable $Z = \frac{(u_1 - u_2)}{2\nu} X$ tending to $\pm\infty$. Any fixed $X \neq 0$, notice $Z \to \pm\infty$ as $\nu \to 0^+$.

3 Time Evolution of a step profile

In the last section we found a steady shock profile. We now show that if we had an initial step profile, *i.e.*

$$u(x, 0) = u_1 \text{ , for } x < 0 \text{ , and } u(x, 0) = u_2 \text{ for } x > 0 \tag{52}$$

for viscous Burger's equation. Then, from Cole-Hopf transformation, we obtain

$$u(x, t) = \frac{\int_{\mathbb{R}} \frac{x-y}{t} \exp\left[-\frac{G(y;x,t)}{2\nu}\right] dy}{\int_{\mathbb{R}} \exp\left[-\frac{G(y;x,t)}{2\nu}\right] dy} \tag{53}$$

$$G(y; x, t) = \int_0^y F(\xi) d\xi + \frac{(x - y)^2}{2t} \tag{54}$$

We note that for $y < 0$,

$$G(y; x, t) = u_1 y + \frac{(y-x)^2}{2t} \tag{55}$$

$$G(y; x, t) = u_2 y + \frac{(y-x)^2}{2t} \tag{56}$$

Then, we obtain

$$\int_{\mathbb{R}} \exp\left[-\frac{1}{2\nu} G(y; x, t)\right] dy = \int_{-\infty}^{0} \exp\left[-\frac{(y-x)^2}{4\nu t} - \frac{u_1 y}{2\nu}\right] + \int_{0}^{\infty} \exp\left[-\frac{(y-x)^2}{4\nu t} - \frac{u_2 y}{2\nu}\right] \tag{57}$$

After some algebra, you write the result as

$$u = u_2 + \frac{u_1 - u_2}{1 + h \exp\left[\frac{u_1 - u_2}{2\nu}(x - ct)\right]} \ , \tag{58}$$

where

$$h = \frac{\int_{-(x-u_2 t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2} d\zeta}{\int_{(x-u_1 t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2} d\zeta} \tag{59}$$

For fixed $\frac{x}{t}$ in the range $u_2 < \frac{x}{t} < u_1$, $h \to 1$ as $t \to \infty$ and the solution approaches the steady solution of the last section.