Week 12 Lecture Notes, Math 6451, Tanveer

1 Introduction to Energy Methods for existence of solution

We now seek to illustrate the energy method to prove existence of solution to evoluationary PDE. We choose the simple case of 1-D Burger's equation for presentational simplicity:

$$u_t - u_{xx} + u u_x = 0 \quad , \qquad x \in (0,1) \ , \qquad t > 0 \ , \tag{1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \ , \qquad , \quad u(0,t) = 0 = u(1,t) \ , \tag{2}$$

where u_0 is a real valued function. With an attempt to avoid heavy mathematical machinery in the proof, we will assume $u_0^{(iv)} \in L_2(0,1)$. This is not necessary; it turns out $u_0' \in L_2(0,1)$ suffices. We will prove the following theorem by energy method

Theorem 1 *If $u_0^{(iv)} \in L_2(0,1)$, for any $T > 0$, there exists classical solution to the initial value problem* (1)-(2)

The proof of this theorem will have to await some preliminary lemmas. The energy method for existence of solution consists of three essential steps:

1. Determine upper bounds on L^2 norms of u and its higher derivative. This process is called *a priori* 'energy' estimates since estimates are based on a u satisfying (1)-(2).

2. Appeal to ODE theory to get local in time solution to a finite dimensional approximation u_n of (1)-(2) satisfying the same energy bounds as u.

3. Appeal to some compactness argument to show that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$ that converges and that the limiting solution u satisfies (1)-(2)

2 *A priori* energy estimate

In this section, we will prove the following proposition

Proposition 2 *If smooth classical solution $u(\cdot,t)$ to* (1)-(2) *exists for $t \in [0,T]$, then if $u_0^{(iv)} \in L^2(0,1)$, then $\|\partial_x^j u\|_{L_2(0,1)} \leq C$ for some constant C independent of t, T and only depends on $\|u_0^{(iv)}\|_{L_2(0,1)}$.*

The proof will involve a lengthy calculation, which we will initiate after some preliminaries. In the following, the notation $(\cdot)$ denotes the $L_2(0,1)$ real valued inner-product and $\|\cdot\|$ denoted the $L_2(0,1)$ norm, while $\|\cdot\|_\infty$ will denote the sup norm in space. The following simple statement is most useful in obtaining energy bounds.

Lemma 3 *(Gronwall): Assume $k(t), g(t) \in \mathbf{C}^0$ and $a(t) \in \mathbf{C}^1$ for $t \geq 0$ and it satisfies*

$$a'(t) \leq k(t)a(t) + g(t) \tag{3}$$

then

$$a(t) \leq \mu(t)a(0) + \int_0^t \frac{\mu(t)}{\mu(\tau)} g(\tau) d\tau \ , \qquad \text{where } \mu(t) = \exp\left[\int_0^t k(\tau)d\tau\right] \tag{4}$$

Proof. Note (3) on multiplication by μ^{-1} implies

$$\frac{a'(t)}{\mu(t)} - \frac{k(t)a(t)}{\mu(t)} \leq \frac{g(t)}{\mu(t)}$$

Then, using the fact that $\frac{d}{dt}\mu^{-1} = -k\mu^{-1}$, it follows that

$$\frac{d}{dt}\left(\frac{a}{\mu}\right) \leq \frac{g(t)}{\mu(t)}$$

Integration leads to

$$\frac{a(t)}{\mu(t)} - \frac{a(0)}{\mu(0)} \leq \int_0^t \mu^{-1}(s)g(s)ds$$

Using $\mu(0) = 1$, we obtain

$$a(t) \leq a(0)\mu(t) + \int_0^t \frac{\mu(t)}{\mu(s)} g(s)ds$$

and the lemma follows. □

Remark 1 *Gronwall's Lemma as stated above does not prevent $a(t) \to -\infty$. However, it is used mostly in situations where $a(t) > 0$ in which case, we have a useful bound on a.*

Lemma 4 *The following statements hold:*

1. *If $v_x \in L^2(0,1)$ and v satisfies $v(0) = 0 = v(1)$, then $\|v\| \leq \|v\|_\infty \leq \|v_x\|$.*
2. *If $v_{xx} \in L^2(0,1)$, with $v(0) = 0 = v(1)$, then $\|v_x\| \leq \|v_x\|_\infty \leq \|v_{xx}\|$.*

PROOF.

$$v(x) = \int_0^x v_x dx \text{ , implies } |v(x)| \leq \int_0^1 |v_x| dx \leq \|v_x\|$$

using Cauchy-Schwartz inequality. Therefore, $\|v\|_\infty \leq \|v_x\|$. Further, from integral expression for $\|\cdot\|$, it follows at once that $\|v\| \leq \|v\|_\infty$. The first part of the Lemma follows.

For the second part, since $v_{xx} \in L_2(0,1)$, from integration by parts it follows that $(\phi_j, v_{xx}) = -\pi^2 j^2 (\phi_j, v) = -\pi^2 j^2 b_j$. Therefore, Parseval's equality applied to the $L_2(0,1)$ basis $\{\phi_j\}_{j=1}^\infty$ implies that

$$\sum_{j=1}^\infty j^4 \pi^4 b_j^2 = \|v_{xx}\|^2$$

Therefore,

$$\|\partial_x v\|_\infty = \|\frac{d}{dx}\sum_{j=1}^\infty b_j \sin(j\pi x)\|_\infty \leq \sum_{j=1}^\infty |b_j| j\pi \leq \frac{1}{\pi}\left(\sum_{j=1}^\infty \frac{1}{j^2}\right)^{1/2} \|v_{xx}\| = \frac{1}{\pi}\sqrt{\frac{\pi^2}{6}}\|v_{xx}\| \leq \|v_{xx}\|$$

From integral expression $\|\cdot\|$, $\|v_x\| \leq \|v_x\|_\infty$. $\square$

Remark 2 *While the fundamental theorem of calculus is being used above in the first part of the proof, if v is not so regular, we can replace a sequence of smooth functions v_n satisfying $v_n(0) = 0 = v_n(1)$ so that $\partial_x v_n$ converges to $\partial_x v$ in the $L^2(0,1)$ sense. We can use the above to show that $\{v_n\}_n$ is a Cauchy sequence in the* sup *norm and hence v is continuous.*

Now, we determine *a priori* energy bounds on u and its derivatives. Taking inner-product of (1) with u and use

$$\int_0^1 u_t u dx = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx \ , \quad \int_0^1 u u_{xx} dx = -\int_0^1 u_x^2 dx \ , \quad \int_0^1 u^2 u_x dx = \left[\frac{u^3}{3}\right]_0^1 = 0$$

to obtain

$$\frac{d}{dt}\frac{1}{2}\|u(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 = 0 \tag{5}$$

Time integration leads to

$$\frac{1}{2}\|u(\cdot,t)\|^2 + \int_0^t \|u_x(\cdot,\tau)\|^2 d\tau = \frac{1}{2}\|u_0\|^2 =: E_0 \tag{6}$$

It follows that each of $\|u(\cdot,t)\|$ and $\int_0^t \|u_x(\cdot,\tau)\|^2 d\tau$ is bounded as long as solution exists. From Lemma 4 and (6), it follows that

$$\int_0^t \|u(\cdot,\tau)\|_\infty^2 d\tau \leq \int_0^t \|u_x(\cdot,\tau)\|^2 d\tau \leq E_0 \tag{7}$$

Taking the x derivative of (1) results in

$$\partial_t u_x - u_{xxx} = -\partial_x(u u_x) \tag{8}$$

Inner product of (8) with u_x, using $u_{xx} = 0$ at $x = 0, 1$ which follows from (1), we obtain on integration by parts

$$\frac{d}{dt}\frac{1}{2}\|u_x\|^2 + \|u_{xx}\|^2 = (u_{xx}, u u_x) \leq \|u(\cdot,t)\|_\infty \|u_x\| \|u_{xx}\| \tag{9}$$

Using $cd \leq \frac{1}{2}c^2 + \frac{1}{2}d^2$, where $c = \|u_{xx}\|$ and $d = \|u(\cdot,t)\|_\infty \|u_x\|$, (9) implies

$$\frac{d}{dt}\frac{1}{2}\|u_x\|^2 + \frac{1}{2}\|u_{xx}\|^2 \leq \frac{1}{2}\|u(\cdot,t)\|_\infty^2 \|u_x\|^2 \tag{10}$$

This implies

$$\frac{d}{dt}\|u_x\|^2 \leq k(t)\|u_x\|^2 \ , \qquad \text{where } k(t) = \|u(\cdot,t)\|_\infty^2 \tag{11}$$

Using Gronwall's lemma on (11), this time with $a(t) = \|u_x\|^2$ and $g(t) = 0$, we obtain from (7),

$$\|u_x(\cdot,t)\|^2 \leq \|u_0'\|^2 \exp\left[\int_0^t k(\tau)d\tau\right] \leq \|u_0'\|^2 e^{E_0} =: E_1 \tag{12}$$

From Lemma 4, applied to $v = u$,

$$\|u(\cdot,t)\|_\infty^2 \leq \|u_x(\cdot,t)\|^2 \leq \|u_0'\|^2 e^{E_0} = E_1 \tag{13}$$

Note maximum principle gives the bound $\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty$. Now, returning to (10) and carrying out time integration, we have

$$\|u_x(\cdot,t)\|^2 + \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau \leq \|u_0'\|^2 + \int_0^t k(\tau)\|u_x(\cdot,\tau)\|^2 d\tau \tag{14}$$

Dropping $\|u_x(\cdot,t)\|^2$ on the left hand side of (14), and using (12) to estimate the right hand side, we obtain

$$\begin{aligned}\int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau &\leq \|u_0'\|^2 + \|u_0'\|^2 \int_0^t k(\tau)\exp\left[\int_0^\tau k(\tau')d\tau'\right] d\tau \\ &\leq \|u_0'\|^2 \exp\left[\int_0^t k(\tau)d\tau\right] \leq \|u_0'\|^2 e^{E_0} = E_1 \end{aligned}\tag{15}$$

However, the derivation (16) solely based on energy argument is signicant since similar ideas are sometimes applicable when maximum principle does not hold. Going back to (14) and ignoring $\|u_x(\cdot,t)\|^2$ on the left side, using (12), we obtain

$$\int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau \leq \|u_0'\|^2 + \|u_0'\|^2 \int_0^t k(\tau)\exp\left[\int_0^\tau k(\tau')d\tau'\right] d\tau$$
$$\leq \|u_0'\|^2 \exp\left[\int_0^t k(\tau)d\tau\right] \leq \|u_0'\|^2 e^{E_0} = E_1 \quad (16)$$

Using second part of Lemma 4, with $v = u$, (16) implies

$$\int_0^t \|u_x(\cdot,\tau)\|_\infty^2 d\tau \leq \|u_0\|^2 e^{E_0} = E_1 \quad (17)$$

Now, consider Energy bounds on $\|u_{xx}\|$. Taking two derivatives of (5) with respect to x, results in

$$u_{xxt} - u_{xxxx} = -\partial_x\left(u_x^2 + uu_{xx}\right) = -3u_xu_{xx} - uu_{xxx} \quad (18)$$

We note that since u, u_{xx} is zero at $x = 0, 1$, it follows from (20) that $u_{xxxx} = 0$ at $x = 0, 1$. Inner-product of (20) with u_{xx}, and using $u_{xx} = 0$ at $x = 0, 1$, integration by parts leads to

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \|u_{xxx}\|^2 = -\left(u_{xx}, 2u_xu_{xx}\right) + \left(u_{xxx}, uu_{xx}\right) = 5\left(u_{xxx}, uu_{xx}\right) \leq 5\|u(\cdot,t)\|_\infty\|u_{xxx}\|\|u_{xx}\| \quad (19)$$

Again using $cd \leq \frac{1}{2}c^2 + \frac{1}{2}d^2$, with $c = u_{xxx}$ and $d = 5\|u(\cdot,t)\|_\infty\|u_{xx}\|$, it follows that

$$5\|u(\cdot,t)\|_\infty\|u_{xx}\|\|u_{xxx}\| \leq \frac{1}{2}\|u_{xxx}\|^2 + \frac{25}{2}k(t)\|u_{xx}\|^2 \quad (20)$$

with $k(t)$ given by (11). Then (19) implies

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \frac{1}{2}\|u_{xxx}\|^2 \leq \frac{25}{2}k(t)\|u_{xx}\|^2 \quad (21)$$

Dropping $\frac{1}{2}\|u_{xxx}\|^2$ on the left of (21) and using Gronwall's Lemma, we have

$$\|u_{xx}\|^2 \leq \|u_0''\|^2 \exp\left[25\int_0^t k(\tau)d\tau\right] \leq \|u_0''\|^2 e^{25E_0} =: E_2 \quad (22)$$

Going back to (21) and doing time integration and using (22), we obtain

$$\int_0^t \|u_{xxx}(\cdot,\tau)\|^2 d\tau \leq \|u_0''\|^2 + 25\int_0^t k(\tau)\|u_{xx}(\cdot,\tau)\|^2 d\tau \leq E_2 \quad (23)$$

From $\|u_x\|_\infty \leq \|u_{xx}\|$, (22) implies

$$\int_0^t \|u_{xx}(\cdot,\tau)\|_\infty^2 d\tau \leq \|u_0''\|^2 e^{25E_0} = E_2 \tag{24}$$

This implies On taking third derivative of (1) with respect to x, we obtain

$$u_{xxxt} - u_{xxxxxx} = \partial_x \left(-3u_x u_{xx} - u u_{xxx}\right) \tag{25}$$

Inner product with u_{xxx}, using $u, u_{xx}, u_{xxxx} = 0$ at the boundary points, we obtain on integration by parts

$$\frac{1}{2}\frac{d}{dt}\|u_{xxx}\|^2 + \|u_{xxxx}\|^2 = (u_{xxxx}, 3u_x u_{xx} + u u_{xxx})$$
$$\leq 3\|u_x\|_\infty \|u_{xxxx}\| \|u_{xx}\| + \|u\|_\infty \|u_{xxx}\| \leq \|u_{xxxx}\|^2 + \frac{9}{2}\|u_x(\cdot,t)\|_\infty^2 \| \|u_{xx}\|^2 + \frac{1}{2}\|u(\cdot,t)\|_\infty^2 \|u_{xxx}\|^2 \tag{26}$$

Using Lemma **??** with $v = u_{xx}$, $\|u_{xx}\| \leq \|u_{xxx}\|$ and therefore (26) implies

$$\frac{d}{dt}\|u_{xxx}\|^2 \leq \left(9\|u_x(\cdot,t)\|_\infty^2 + \|u(\cdot,t)\|_\infty^2\right) \|u_{xxx}\|^2 \tag{27}$$

Using Gronwall's lemma, (7) and (17), it follows that

$$\|u_{xxx}(\cdot,t)\|^2 \leq \|u_0'''\|^2 \exp\left[\int_0^t \tilde{k}(\tau) d\tau\right] , \tag{28}$$

where

$$\tilde{k}(t) = 9\|u_x(\cdot,t)\|_\infty^2 + \|u(\cdot,t)\|_\infty^2 . \tag{29}$$

Therefore, using bounds on each term constiuting $\int_0^t \tilde{k}(\tau) d\tau$ (28) implies

$$\|u_{xxx}(\cdot,t)\|^2 \leq \|u_0'''\|^2 \exp\left[\int_0^t \tilde{k}(\tau) d\tau\right] \leq \|u_0'''\|^2 e^{9E_2 + E_0} =: E_3 \tag{30}$$

This implies that

$$\|u_{xx}(\cdot,t)\|_\infty^2 \leq E_3 \tag{31}$$

Now, going back to (26) and integrating from 0 to t, we have

$$\|u_{xxx}(\cdot,t)\|^2 + \int_0^t \|u_{xxxx}(\cdot,\tau)\|^2 d\tau \leq \|u_0'''\|^2 + \int_0^t \tilde{k}(\tau)\|u_{xxx}(\cdot,\tau)\|^2 d\tau \tag{32}$$

Dropping $\|u_{xxx}(\cdot,t)\|^2$ on the left hand side of (32), and using (30), we have

$$\int_0^t \|u_{xxxx}(\cdot,\tau)\|^2 d\tau \leq \|u_0'''\|^2 e^{9E_2+E_0} =: E_3 \tag{33}$$

We note taking an additional derivative of (25) with respect to x gives

$$\partial_t[\partial_x^4 u] - \partial_x^6 u = -\partial_x \left(3u_{xx}^2 + 4u_x u_{xxx} + u u_{xxxx}\right) \tag{34}$$

Therefore, on inner-product with $\partial_x^4 u$, and integration by parts using $u, u_{xx}, u_{xxxx} = 0$ at $x = 0, 1$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x^4 u\|^2 + \|\partial_x^5 u\|^2 &= \left(\partial_x^5 u, 3(\partial_x^2 u)^2 + 4\partial_x u \partial_x^3 u + u\partial_x^4 u\right) \\
&\leq \frac{1}{2}\|\partial_x^5 u\|^2 + \left\{3\|u_{xx}(\cdot,t)\|_\infty \|\|u_{xx}\| + +4\|u_x(\cdot,t)\|_\infty \|u_{xxx}\| + \|u(\cdot,t)\|_\infty \|\partial_x^4 u\|\right\} \\
&\leq \frac{1}{2}\|\partial_x^5 u\|^2 + c_0\|u_{xx}(\cdot,t)\|_\infty^2 \|u_{xx}\|^2 + c_1\|u_x(\cdot,t)\|_\infty^2 \|u_{xxx}\| + c_2\|u(\cdot,t)\|_\infty^2 \|\partial_x^4 u\|^2 \\
&\leq \frac{1}{2}\|\partial_x^5 u(\cdot,t)\|^2 + \frac{1}{2}k_1(t)\|\partial_x^4 u\|^2
\end{aligned} \tag{35}$$

where

$$k_1(t) = 2c_0\|u_{xxx}(\cdot,t)\|^2 + 2c_1\|u_{xx}(\cdot,t)\|^2 2c_2\|u_x(\cdot,t)\|^2 \tag{36}$$

for constants c_0, c_1 and c_2 independent of t, where we used $\|u_{xx}\| \leq \|u_{xxx}\| \leq \|\partial_x^4 u\|$.

$$\frac{d}{dt}\|\partial_x^4 u(\cdot,t)\|^2 + \|\partial_x^5 u(\cdot,t)\|^2 \leq k_1(t)\|\partial_x^4 u\|^2 \tag{37}$$

It follows from (37) that

$$\frac{d}{dt}\|\partial_x^4 u\|^2 \leq k_1(t)\|\partial_x^4 u\|^2 \tag{38}$$

Therefore, using bounds (7), (17), (**??**), we obtain from Gronwall's lemma applied to (38)

$$\|\partial_x^4 u\|^2 \leq \|u_0^{(iv)}\|^2 \exp\left[\int_0^t k_1(\tau) d\tau\right] \tag{39}$$

Using (7), (17) we obtain

$$\|\partial_x u\|^2 \leq \|u_0^{(iv)}\|^2 \exp\left[2c_0 E_2 + 2c_1 E_1 + 2c_2 E_0\right] =: 2E_4 \ , \tag{40}$$

Going back to (37), and integrating in t, and using (39), we also obtain

$$\int_0^t \|\partial_x^5 u(\cdot,\tau)\|^2 d\tau \leq E_4 \tag{41}$$

3 Galerkin Approximation and Energy Estimates

We now introduce a finite-dimensional approximation of (1)-(2). Define $\phi_j(x) = \sqrt{2}\sin(j\pi x)$. We note that $\{\phi_j\}_{j=1}^{\infty}$ forms an orthonormal basis for functions in $L_2(0,1)$. We define $\mathcal{S}_n \subset L_2(0,1)$ a finite dimensional subspace

$$\mathcal{S}_n = \left\{ v \in L_2(0,1) : v(x) = \sum_{j=1}^{n} b_j \phi_j(x) \right\}$$

We define projection operator $\mathcal{P}_n$ from $L_2(0,1)$ to $\mathcal{S}_n$ so that if $v \in L^2(0,1)$ with

$$v = \sum_{j=1}^{\infty} v_j(t)\phi_j(x) \ , \qquad \text{, then} \ \ \mathcal{P}_n v = \sum_{j=1}^{n} v_j(t)\phi_j(x) \tag{42}$$

Note that $\|\mathcal{P}_n v\| \leq \|v\|$, the equality holding for $v \in \mathcal{S}_n$. Here is a preliminary lemma that will prove useful later.

Lemma 5 *For any integer $k \geq 0$, if $v \in \mathcal{S}_n$ then, for any $w \in L_2(0,1)$, we have*

$$\left(\partial_x^k v, \partial_x^k \mathcal{P}_n w\right) = \left(\partial_x^k v, \partial_x^k w\right) \tag{43}$$

Proof. For $k = 0$, we note from orthogonality of $\{\phi_j(x)\}_{j=1}^{\infty}$ that

$$(v, \mathcal{P}_n w) = \sum_{j=1}^{n} v_j w_j = (v, w)$$

If k is even, then it is clear from defining of $\mathcal{P}_n$ that ∂_x^k and $\mathcal{P}_n$ commute and we obtain

$$\left(\partial_x^k v, \mathcal{P}_n[\partial_x^k w]\right) = \left(\partial_x^k v, \partial_x^k w\right)$$

Using the result for $k = 0$, this time with w replaced by $\partial_x^k w$ and v replaced by $\partial_x^k v$, the lemma follows. If k is odd, using even derivatives of v to be zero at $x = 0, 1$, on integration by parts,

$$\left(\partial_x^k v, \partial_x^k \mathcal{P}_n w\right) = -\left(\partial_x^{(k-1)} v, \partial_x^{(k+1)} \mathcal{P}_n w\right)$$
$$= -\left(\partial_x^{(k-1)} v, \mathcal{P}_n \partial_x^{(k+1)} w\right) = -\left(\partial_x^{(k-1)} v, \partial_x^{(k+1)} w\right) = \left(\partial_x^k v, \partial_x^k w\right)$$

□

Lemma 6 *If $w_n \to w$ in $\mathbf{C}^1[0,1]$, then $\|\mathcal{P}_n w_n - w\|_\infty \to 0$ as $n \to \infty$.*

PROOF. Clearly $w, \partial_x w \in L^2$; therefore there exists N so so that $\|(I - \mathcal{P}_N)w\|_\infty \leq \|\partial_x (I - \mathcal{P}_N) w\| \to 0$ as $N \to \infty$. Now, we have

$$\|\mathcal{P}_N(w_n - w)\|_\infty \leq \|\partial_x \mathcal{P}_n (w_n - w)\| \leq \|\partial_x (w_n - w)\| \to 0$$

□ In the spirit of numerical computation, we define a Galerkin approximation to (1)-(2) by seeking solution $u_n \in \mathcal{S}_n$ with a finite representation

$$u_n(x,t) = \sum_{j=1}^{n} a_j(t) \phi_j(x) \tag{44}$$

with a_j chosen to satisfy

$$\partial_t u_n - \partial_{xx} u_n = -\mathcal{P}_n \left[u_n \partial_x u_n \right] \tag{45}$$

$$u_n(x,0) = \mathcal{P}_n u_0 \tag{46}$$

This is equivalent to the set of finite nonlinear ODEs for $j = 1, ...n$:

$$\frac{d}{dt} a_j = -j^2 \pi^2 a_j - \pi \left(\phi_j, \sum_{m=1}^{n} \sum_{k=1}^{n} k \left\{ a_m a_k \phi_{m+k} + a_m a_k \phi_{m-k} \right\} \right) \tag{47}$$

with initial condition

$$a_j(0) = (\phi_j, u_0) \tag{48}$$

Note that we write (45)-(46) in the integral form

$$u_n(x,t) = \mathcal{P}_n u_0 + \int_0^t \left\{ \partial_x^2 u_n(x,\tau) - \mathcal{P}_n \left[u_n \partial_x u_n \right](x,\tau) \right\} d\tau \tag{49}$$

We will prove that

Proposition 7 *For any integer n, for any $T > 0$, there exists unique solution to the ODE system* (47)-(48) *for $t \in [0,T]$ implying unique solution u_n satisfying* (45)-(46).

PROOF. For any fixed $n \geq 1$, the proof of local existence of solution of the ODE system (47)-(48) follows from ODE theory for interval $t \in [0, T_n]$. However, it is also known from ODE theory that if solution $\mathbf{a}(t) = \{a_j(t)\}_{j=1}^n$ remains bounded as $t \to T_n^-$, then the solution can be continued beyond $t = T_n^-$. Now, since (47)-(48) is equivalent to (45)-(46), we multiply (45) by u_n and integrate by parts to obtain

$$\frac{d}{dt} \frac{1}{2} \|u_n\|^2 + \|\partial_x u_n\|^2 = 0$$

Here we used

$$(u_n, \mathcal{P}_n[u_n \partial_x u_n]) = (u_n, u_n \partial_x u_n) = \frac{1}{3} \int_0^1 \partial_x(u_n^3) dx = 0$$

Therefore, it follows that

$$\|u_n(\cdot, t)\|^2 \leq \|P_n u_0\|^2 \leq \|u_0\|^2 = E_0$$

Therefore, from Parseval's inequality,

$$\sum_{j=1}^{n} a_j^2 \leq E_0$$

where E_0 is independent of t and therefore blow up of $\mathbf{a}(t)$. Hence the ODE solution $\mathbf{a}(t)$ can be continued indefinitely. This implies u_n satisfies (45)-(46). □

Lemma 8 *For any integer n, the solution u_n found in Propositon 7 for satisfies the same bounds as the a priori bounds on u, i.e. for $j = 0, \cdots, 4$,*

$$\|\partial_x^j u_n\| \ , \tag{50}$$

where C is independent of n and T and only depends on L_2 bounds of $u_0^{(iv)}$. Furthermore,

$$\int_0^t \|\partial_x^j u_n\,(\cdot, t)\,\|^2 \leq C \tag{51}$$

for $j = 1, \cdot, 5$, where C is independent of n and T and only depends on L_2 bounds of $u_0^{(iv)}$.

Proof. The proof is very similar to calculation of *a priori* bounds on u, except that we will need to use Lemma 5 to rid ourselves of $\mathcal{P}_N$ in the inner product. Note we use the same procedure to find time integral estimates of u_n as we did for u in determining (7), (16), (24), (33), (41) since P_n drops out in the L_2 inequalities. The details are left as an exercise. □

Lemma 9 *For any integer $n \geq 1$, the solution u_n found in Propositon 7 for also satisfies*

$$\|\partial_t u_n\| \ , \|\partial_{xxt} u_n\|, \|\partial_{tt} u_n\| \leq C \tag{52}$$

with C independent of n and T, while for $t \in [0, T]$, while

$$\int_0^t \|\partial_{ttx} u_n\,(\cdot, \tau)\,\|^2 d\tau \leq C_3(T) \tag{53}$$

PROOF. As far as controlling the t derivatives of u_n We note first that

$$\partial_t u_n = \partial_{xx} u_n - P_n \left[u_n \partial_x u_n \right] \tag{54}$$

and therefore taking L^2 estimates and realizing that P_n is irrelevant in the inequality, we have

$$\|\partial_t u_n\| \le \|\partial_{xx} u_n\| + \|u_n\|_\infty \|\partial_x u_n\| \le C \ , \tag{55}$$

since each term has a bound as given, noticing in particular $\|u_n\|_\infty \le \|\partial_x u_n\|$. Moreover, we note that

$$\partial_t \partial_x u_n = \partial_{xxx} u_n - \partial_x P_n \left[\partial_x u_n \partial_x u_n \right] \tag{56}$$

and so taking L^2 estimates and realizing that P_n is irrelevant in the inequality, we have

$$\|\partial_t \partial_x u_n\| \le \|\partial_{xxx} u_n\| + \|\partial_x u_n\|_\infty \|\partial_x u_n\| + \|u_n\|_\infty \|\partial_{xx} u_n\| \le C \tag{57}$$

noticing that $\|\partial_x u_n\|_\infty \le \|\partial_{xx} u_n\|$. Also, note that

$$\partial_{xxt} u_n = \partial_{xxxx} u_n - \partial_{xx} P_n \left[u_n \partial_x u_n \right] \tag{58}$$

So, it follows that

$$\|\partial_{xxt} u_n\| \le \|\partial_{xxxx} u_n\| + 3\|\partial_x u_n\|_\infty \|\partial_{xx} u_n\| + \|u_n\|_\infty \|\partial_{xxx} u_n\| \le C \tag{59}$$

Now, taking an additional x derivative, we have

$$\partial_{xxxt} u_n = \partial_x^5 u_n - \partial_{xxx} P_n \left[u_n \partial_x u_n \right] \tag{60}$$

and so

$$\|\partial_{xxxt} u_n\| \le \|\partial_x^5 u_n\| + 3\|\partial_{xx} u_n\|_\infty \|\partial_{xx} u_n\| + 4\|\partial_x u_n\|_\infty \|\partial_{xxxx} u_n\| \le C + \|\partial_x^5 u_n\| \tag{61}$$

From the Burger's equation, we

$$\partial_{xtt} u_n = \partial_{xxxt} u_n - \partial_x P_n \left[\partial t u_n \partial_x u_n \right] - -\partial_x P_n \left[u_n \partial_{xt} u_n \right] \tag{62}$$

It follows that

$$\|\partial_{xtt} u_n\| \le \|\partial_{xxxt} u_n\| + 2\|\partial_x u_n\|_\infty \|\partial_{xt} u_n\| + \|\partial_x u_n\|_\infty \|\partial_{xt} u_n\|_\infty + \|u_n\|_\infty \|u_{xxt}\| \le C + \|\partial_x^5 u_n\| \tag{63}$$

So, it follows that[1]

$$\int_0^t d\tau \|\partial_x u_{tt}(\cdot,\tau)\|^2 \leq \int_0^t d\tau \left(C + \|\partial_x^5 u_n(\cdot,t)\|\right)^2$$
$$\leq \hat{C}_3(T)\left(1 + \int_0^t \|\partial_x^5 u_n(\cdot,\tau)\|^2 d\tau\right) \leq C_3(T) \quad (64)$$

□

Lemma 10 *For any fixed x, $\{\partial_t u_n(x,t)\}_{n=1}^{\infty}$ is an equicontinuous family of functions of $t \in [0,T]$ and therefore has a subsequence that converges uniformly for $t \in [0,T]$*

Proof. We note that it is enough to show $\int_0^T \left(\partial_{tt} u_n(x,t)\right)^2 < C$, for some constant C independent of n since for $T \leq t_2 > t_1 \geq 0$,

$$\left|\partial_t u_n(x,t_2) - \partial_t u_n(x,t_1)\right| \leq \int_{t_1}^{t_2} \left|\partial_{tt} u_n(x,\tau)\right| d\tau \leq C^{1/2}\sqrt{t_2 - t_1}$$

However, we have for each x, $\left|\partial_{tt} u_n(x,t)\right| \leq \|\partial_x \partial_{tt} u_n(\cdot,t)\|$ and hence

$$\int_0^T \left(\partial_{tt} u_n(x,t)\right)^2 dt \leq \int_0^T \|\partial_{xtt} u_n(\cdot,t)\|^2 dt \leq C_3(T)$$

from previous lemma, and hence $\{\partial_t u_n(x,t)\}_{n=1}^{\infty}$ is equicontinuous in $t \in [0,T]$ for each x. Therefore, it must have a convergent subsequence that converges uniformly for $t \in [0,T]$.
□

Lemma 11 *Define $\mathbf{v}_n = (u_n, \partial_x u_n, \partial_{xx} u_n)$. Then for $t \in [0,T]$, exists a subsequence $\left\{\mathbf{v}_{n_j}\right\}_{j=1}^{\infty}$ which converges in the sup norm to $\mathbf{v} = (u, u_x, u_{xx})$, where u satisfies the integral equation*

$$u(x,t) = u_0(x) + \int_0^t \left\{\partial_x^2 u(x,\tau) - [u u_x](x,\tau)\right\} d\tau$$

implying u satisfies (1)-(2)

Proof. Since $\|\partial_x^j u_n\|$ for $j = 0, \cdots 4$ have bounds independent of t and n, it follows that $\|\partial_x^j u_n\|_\infty$ for $j = 0, \cdots 3$ has bounds independent of t and n. Therefore $\{\mathbf{v}_n\}_{n=1}^{\infty}$ forms an

[1] A more refined estimate will have shown that C_3 does not depend on T, but for our purposes, what is shown is enough

equi-continuous family of functions which has a subsequence $\mathbf{v}_{n_j}$ that converges in the sup norm for each $t \in [0, T]$ to $\mathbf{v}$. zero at $x = 0, 1$. Using the form of $\mathbf{v}_n$, it is not difficult to prove that $\mathbf{v} = (u, u_x, u_{xx})$ for some scalar function u. We now prove that u satisfies Burgers equation. Since $u_{n_j} \to u$ in $\mathbf{C}^3(0,1)$ for each $t \in [0, T]$, it follows that $\|u_{n_j}\partial_x u_{n_j} - u\partial_x u\|_\infty$, $\|\partial_x \left\{u_{n_j}\partial_x u_{n_j} - u\partial_x u\right\}\|_\infty \to 0$ as $j \to \infty$. Also each term in the time integral

$$\int_0^t \left\{\partial_x^3 u_{n_j} - \mathcal{P}_{n_j}(u_{n_j}\partial_x u_{n_j})\right\}(x,\tau)d\tau$$

is bounded independent of t in the $\sup_{x\in[0,1]}$ sense. Furthermore, using previous lemma, for any given x, a subsequence of this $\left\{u_{n_j}\right\}_{j=1}^{\infty}$, which with slight abuse of notation is still denoted by $u_{n_j}(x,t)$ converges uniformly for $t \in [0, T]$ as $j \to \infty$. Therefore, from dominating convergence theorem, and using Lemma 6, we have

$$\lim_{j\to\infty} \int_0^t \left\{\partial_x^3 u_{n_j} - \mathcal{P}_{n_j}(u_{n_j}\partial_x u_{n_j})\right\}(x,\tau)d\tau = \int_0^t \left\{\partial_x^3 u - u\partial_x u)\right\}(x,\tau)d\tau$$

Further, it is clear that $\|\mathcal{P}_n u_0 - u_0\|_\infty \to 0$. On the other hand, $\lim_{j\to\infty} u_{n_j}(x,t) = u(x,t)$. It follows from (49) that u satisfies

$$u(x,t) = u_0(x) + \int_0^t \left\{\partial_x^2 u - u\partial_x u\right\}(x,\tau)d\tau$$

which immediately implies $u(x,t)$ satisfies Burgers equation with initial condition $u_0(x)$. □ This completes the proof of Theorem1

4 Blow-up of solution to PDE

Consider

$$u_t - \Delta u = u^2 \text{ for } \mathbf{x} \in \mathbb{R}^n, t > 0 \ , \quad u(x,0) = F(x) \tag{65}$$

We can prove existence of solution for $[0, T]$ for T small enough in the same way as we did for viscous Burger's equation. However, unlike the viscous Burger's equation, there is no maximum principle for this problem, though it satisfies minimum principe. In particular if $F > 0$, then $u > 0$ as long as a smooth classical solution exists that is bounded at ∞. Indeed, as we will prove now, solution in this class will blow up in finite time if $F > 0$. For that purpose define $G(x,t)$

$$G(\mathbf{x},t) = \frac{1}{[4\pi(T_1 - t)]^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4(T_1 - t)}\right] \tag{66}$$

Note that $G(\mathbf{x},t)$ is a smooth solution of backwards heat equation for $t \in (0, T_1)$:

$$G_t = -\Delta G \tag{67}$$

We will choose T_1 large enough so that

$$T_1 > \left(\int_{\mathbb{R}^n} G(\mathbf{x},0) F(\mathbf{x}) d\mathbf{x} \right)^{-1} \tag{68}$$

This can always be arranged for $F(\mathbf{x}) > F_m > 0$ since the right hand side of (68) $\le \frac{1}{F_m}$ Using (67) and (65), it follows on using Green's identity that for functions u that are bounded at ∞,

$$\frac{d}{dt} \int_{\mathbb{R}^n} u(\mathbf{x},t) G(\mathbf{x},t) d\mathbf{x} = \int_{\mathbb{R}^n} G(\mathbf{x},t) u^2(\mathbf{x},t) d\mathbf{x} \tag{69}$$

If $F(\mathbf{x}) > 0$, any classical solution $u(\mathbf{x},t) > 0$ within the existence time. Then, we also note that using Cauchy-Schwartz inequality

$$\int_{\mathbb{R}^n} G(\mathbf{x},t) u(\mathbf{x},t) d\mathbf{x} \le \left\{ \int_{\mathbb{R}^n} G(\mathbf{x},t) u^2(\mathbf{x},t) d\mathbf{x} \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} G(\mathbf{x},t) d\mathbf{x} \right\}^{1/2} = \left\{ \int_{\mathbb{R}^n} G(\mathbf{x},t) u^2(\mathbf{x},t) d\mathbf{x} \right\}^{1/} \tag{70}$$

Therefore, it follows from (69) that if we define

$$y(t) = \int_{\mathbb{R}^n} u(\mathbf{x},t) G(\mathbf{x},t) d\mathbf{x} \ , \tag{71}$$

then (70) implies that

$$\frac{dy}{dt} \ge y^2 \ , y(0) = y_0 := \int_{\mathbb{R}^n} G(\mathbf{x},0) F(\mathbf{x},0) > 0 \tag{72}$$

Then

$$y(t) \ge \frac{y_0}{(1 - t y_0)} \tag{73}$$

which blows up at $t = T_c \le \frac{1}{y_0}$. Therefore, diffusion is not strong enough to prevent finite time blow up.