# Week 14 Lecture Notes, Math 6451, Tanveer

## 1 Fourier Approach to contraction mapping

Consider once again Burger's equation with

$$u_t + uu_x = u_{xx} \ , u(x,0) = F(x) \ \text{ for } x \in \mathbb{R} \tag{1}$$

We will assume the Fourier Transform of initial condition $F$ exists and is in $L^1(\mathbb{R})$ and we will prove that classical solutions to (1) exists for $t \in (0,T]$ for sufficiently small $T$. Since this solution is smooth for $t \in (0,T]$ and in particular $u(\cdot,T)$ is continuous, we can use previous approach in the physical domain to march from $t \in [T,2T]$, and then use maximum principle to show global existence. Note the Fourier approach uses a different assumption on initial condition than what we have seen before. This approach is particularly efficient in showing instantaneous smoothness of solution for $t$ small but positive and is a general enough as a method to be applicable in many complicated problems like Navier-Stokes PDE. We will start by formally carrying out Fourier Transform in $x$. We will then show that the transformed equation has a solution in a space of functions that guarantees, its inverse Fourier transform satifies (1) There is also other advantages to this approach, including easy extension to higher space dimensions. We define

$$\hat{u}(k,t) = \mathcal{F}\left[u(\cdot,t)\right](k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x,t) dx \tag{2}$$

We also note that if Fourier Transform of $u_{xx}$, $[u^2]_x$, $u_t$ were to exist, then

$$\mathcal{F}\left[u_t(\cdot,t)\right] = \hat{u}_t(k,t) \ , \tag{3}$$

$$\mathcal{F}\left[u_{xx}(\cdot,t)\right] = -k^2 \hat{u}(k,t) \ , \tag{4}$$

$$\mathcal{F}\left[(u^2))_x(\cdot,t)\right] = ik \left[\hat{u} * \hat{u}\right](k,t) \ , \tag{5}$$

where the Fourier Convolution $*$ is defined by

$$\left[\hat{u} * \hat{u}\right](k,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\eta,t)\hat{u}(k-\eta,t) d\eta \ . \tag{6}$$

Note if indeeed $\hat{u}(\cdot,t) \in L^1(\mathbb{R})$, the by using Fubini's theorem, we can directly prove from expression (6) that

$$\mathcal{F}^{-1}\left[\hat{u} * \hat{u}\right](x,t) == \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \int_{\mathbb{R}} \hat{u}(k-\eta,t)\hat{u}(\eta,t) d\eta dk = u^2(x,t) \tag{7}$$

which justifies (5) with some assumption on $\hat{u}$. At this stage, since all the steps are formal, we don't worry about the assumptions. Using (3)-(5) in (1), we formally obtain

$$\hat{u}_t + k^2 \hat{u} = -ik\hat{u} * \hat{u} \ , \quad \hat{u}(k,0) = \hat{F}(k) \tag{8}$$

Using method of integrating factor (8) for first order ODEs, (8) is equivalent to

$$\hat{u}(k,t) = \hat{u}^{(0)}(k,t) - ik \int_0^t e^{-k^2(t-\tau)} \left[ \hat{u} * \hat{u} \right](k,\tau) d\tau =: \mathcal{N}\left[\hat{u}\right](k,t) \ , \tag{9}$$

where

$$\hat{u}^{(0)}(k,t) = \hat{F}(k) e^{-k^2 t} \ . \tag{10}$$

**Definition 1** *For $t \in [0,T]$, we define $\mathcal{S}$ as functions in $t$ for each $k$, that is integrable in $k$ with norm*

$$\|\hat{u}\| = \int_{\mathbb{R}} \sup_{(0,T]} \left| \hat{u}(k,\cdot) \right| dk \tag{11}$$

**Lemma 2** *$u^{(0)} \in \mathcal{S}$ with*

$$\|u^{(0)}\| \leq \|\hat{F}\|_{L^1} \tag{12}$$

Proof. We note from expression (10) that since $\hat{F}(k)$ exists almost everywhere

$$\sup_{(0,T]} \left| \hat{u}^{(0)}(k,t) \right| \leq |\hat{F}(k)| \tag{13}$$

and hence from definition of $\| \cdot \|$, the Lemma follows. □

**Theorem 3** *If*

$$T < \frac{\pi}{8 C_0^2 \|\hat{F}\|_{L^1}^2} \quad , \text{ where } C_0 = \sup_{\xi \geq 0} \frac{1 - e^{-\xi}}{\sqrt{\xi}} \ , \tag{14}$$

*$\mathcal{N}$ defined in (9) maps the closed ball $B_{2\|\hat{F}\|_{L^1}} \subset \mathcal{S}$ back to itself and is contractive, implying that (9) has a unique solution $\hat{u}(k,t)$ in the space $\mathcal{S}$ for $t \in [0,T]$.*

Proof. We define cumulative function $w(k,t)$ through

$$w(k,t) = \sup_{\tau \in [0,t]} \left| \hat{u}(k,\tau) \right| \tag{15}$$

We note that for $\tau \in [0,t]$, for any $k$,

$$\left| \left[ \hat{u} * \hat{u} \right](k,\tau) \right| \leq \int_{\mathbb{R}} \left| \hat{u}(\eta,\tau) \right| \left| \hat{u}(k-\eta,\tau) \right| d\eta \leq \int_{\mathbb{R}} w(\eta,t) w(k-\eta,t) d\eta = \left[ w * w \right](k,t) \tag{16}$$

and therefore, it follows from (9) that for $t \in [0,T]$

$$\Big| -ik \int_0^t e^{-k^2}(t-\tau) \left[\hat{u} * \hat{u}\right](k,\tau) d\tau \Big| \leq \left[w * w\right](k,t) \int_0^t e^{-k^2(t-\tau)} |k| d\tau$$
$$\leq \sqrt{T} \left[w * w\right](k,T) \sup_{\xi \geq 0} \frac{1-e^{-\xi}}{\sqrt{\xi}} \leq C_0 \sqrt{T} \left[w * w\right](k,T) \quad (17)$$

Noting that $\|\hat{u}\| = \|w(\cdot,T)\|_{L^1(\mathbb{R}}$ and that for $w(\cdot,t)$ in $L^1$,

$$\|w(\cdot,T) * w(\cdot,T)\|_{L^1} = \frac{1}{\sqrt{2\pi}} \|w(\cdot,T)\|_{L^1}^2 \quad (18)$$

we obtain from (9), (17) and Lemma 2, with given restriction on $T$,

$$\|\mathcal{N}[\hat{u}]\| \leq \|\hat{F}\|_{L^1} + \frac{C_0\sqrt{T}}{\sqrt{2\pi}} \|\hat{u}\|^2 \leq 2\|\hat{F}\|_{L^1} \quad (19)$$

In an anologous sequence of steps, we also have

$$\|\mathcal{N}[\hat{u}_1] - \mathcal{N}[\hat{u}_2]\| \leq \frac{C_0\sqrt{T}}{\sqrt{2\pi}} \left(\|\hat{u}_1\| + \|\hat{u}_2\|\right) \|\hat{u}_1 - \hat{u}_2\| \leq \alpha \|\hat{u}_1 - \hat{u}_2\| \ , \quad (20)$$

where

$$\alpha = \frac{4C_0\sqrt{T}}{\sqrt{2\pi}} \|\hat{F}\|_{L^1} < 1 \quad (21)$$

This proves the contractivity of $\mathcal{N}$ in the ball $B_{2\|\hat{F}\|_{L_1}} \subset \mathcal{S}$ and from Banach fixed point theorem, there exists unique solution $\hat{u}$ to (9) in that ball. □

**Remark 1** *Though the uniqueness of solution has only been proved in a ball $B$, any other solution continuous in time in the space $\mathcal{S}$ must be in this ball for $T$ is small enough as it contains the initial condition $\hat{F}$.*

**Lemma 4** *For $\epsilon > 0$, $t \in [\epsilon, T]$, the solution $\hat{u}$ in Theorem 3, $k\hat{u}(\cdot,t)$, $k^2\hat{u}(\cdot,t)$ are each in $l^1$ as is $k\left[\hat{u} * \hat{u}\right](\cdot,t)$, with uniform bounds for $t \in [\epsilon, T]$.*

Proof. For any $\epsilon > 0$ small, consider $t \in [\epsilon, T]$. Define

$$\hat{w}_\epsilon(k) = \sup_{t \in [\epsilon,T]} \Big|\hat{u}(k,t)\Big| \quad (22)$$

We note that on this time interval

$$\left|k\hat{u}^{(0)}(k,t)\right| \leq \left|\hat{F}(k)\| \sup_{t\in[\epsilon,T]} \left(|k|e^{-k^2 t}\right) \leq \epsilon^{-1/2} \left(\sup_{\xi\geq 0} \sqrt{\xi} e^{-\xi}\right) |\hat{F}(k)| \ , \tag{23}$$

implying at once that $k\hat{u}^{(0)}(\cdot,t) \in L^1$. The same argument as above shows that

$$\left|k^2\hat{u}^{(0)}(k,t)\right| \leq \left|\hat{F}(k)\| \sup_{t\in[\epsilon,T]} \left(|k|^2 e^{-k^2 t}\right) \leq \epsilon^{-1} \left(\sup_{\xi\geq 0} \xi e^{-\xi}\right) |\hat{F}(k)| \ , \tag{24}$$

Therefore, we have from (9)

$$\left|k\hat{u}(k,t)\right| \leq \left|k\hat{u}^{(0)}(k,t)\right| + [\hat{w}_\epsilon * \hat{w}_\epsilon] \int_0^t k^2 e^{-k^2(t-\tau)} d\tau \leq \left|k\hat{u}^{(0)}(k,t)\right| + C_1 \hat{w}_\epsilon * \hat{w}_\epsilon \ , \tag{25}$$

implying at once that

$$w_\epsilon^1(k) := \sup_{t\in[\epsilon,T]} \left|k\hat{u}(\cdot,t)\right| \in L^1 \tag{26}$$

Now, we note on replacing integration variable $\eta \to (k-\eta)$ that

$$\int_{\mathbb{R}} d\eta \eta \hat{u}(k-\eta)\hat{u}(\eta) = \int_{\mathbb{R}} d\eta (k-\eta)\hat{u}(k-\eta)\hat{u}(\eta) \tag{27}$$

and therefore

$$k\hat{u} * \hat{u} = k\int_{\mathbb{R}} d\eta \eta \hat{u}(k-\eta)\hat{u}(\eta) = \int_{\mathbb{R}} d\eta (k-\eta)\hat{u}(k-\eta)\hat{u}(\eta) + \int_{\mathbb{R}} d\eta \eta \hat{u}(k-\eta)\hat{u}(\eta)$$
$$= 2\hat{u} * (k\hat{u}) \tag{28}$$

Therefore, it follows that since each of $\hat{u}$ and $k\hat{u}$ is in $L^1$,

$$\mathcal{F}^{-1}\left[ik\hat{u} * \hat{u}\right](x) = \left(\mathcal{F}^{-1}\left[\hat{u}\right](x)\right)\left(\mathcal{F}^{-1}\left[ik\hat{u}\right](x)\right) = uu_x \tag{29}$$

Using (28) in (9), we have

$$k^2\hat{u}(k,t) = k^2\hat{u}^{(0)}(k,t) - ik^2 \int_0^t e^{-k^2(t-\tau)} \left[(k\hat{u}) * \hat{u}\right](k,\tau) d\tau \ , \tag{30}$$

from which it follows that for $t \in [\epsilon, T]$, we have

$$\left|k^2\hat{u}(k,t)\right| \leq C_2\epsilon^{-1}\left|F(k)\right| + C_1\left[w_\epsilon^1 * w_\epsilon\right] \tag{31}$$

Therefore, $k^2\hat{u}(\cdot,t) \in L^1$ for $t \in [\epsilon, T]$ with

$$\|k^2\hat{u}(\cdot,t)\|_{L^1} \leq C_2\epsilon^{-1}\|F\|_{L^1} + C_1\|w_\epsilon^1\|_{L^1}\|w_\epsilon\|_{L^1} \tag{32}$$

□

**Proposition 5** *We define*

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{u}(\cdot,t)\right](x) \ , \tag{33}$$

*where $\hat{u}(k,t)$ is the solution to (9) found in Theorem 8. $u(x,t)$ is a classical solution to viscous Burgers equation for $t \in (0,T]$.*

PROOF. Note for $\hat{u}(k,t) \in \mathcal{S}$, (9) is equivalent tto the initial value problem

$$\hat{u}_t = -k^2 \hat{u} - ik\hat{u} * \hat{u} \ , \tag{34}$$

with

$$\hat{u}(k,0) = \hat{F}(k) \tag{35}$$

Now, every term on the right hand side of (34) is in $L^1$, as must therefore be $\hat{u}_t$. On inverse Fourier-Transform we get continous $u_{xx}$, $uu_x$ and $u_t$, which from (34)-(35) satisfies

$$u_t = u_{xx} - uu_x \quad , \quad u(x,0) = F(x) \tag{36}$$

Note that uniform $l^1$ bounds of $\hat{u}_t(\cdot,t)$, dependent of $t \in [\epsilon, T]$ is what allows to claim

$$u_t(x,t) = \lim_{h\to 0} \frac{u(x,t+h) - u(x,t)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \frac{\hat{u}(k,t+h) - \hat{u}(k,t)}{h} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}_t(k,t) e^{ikx} \tag{37}$$

□

# 2 Contraction Mapping application in other problems

Contraction mapping Theorem is quite powerful in a large number of problems. As another example, consider for instance the following nonlinear elliptic problem

$$\Delta u = u^2 \ \text{ for } \mathbf{x} \in \Omega \subset \mathbb{R}^n \quad \text{with } u(\mathbf{x}) = h \ \text{ on } \partial\Omega \tag{38}$$

where we will assume $\Omega$ is bounded and has appropriate smoothness for existence of Greens Function for Laplacian. Then inverting Laplacian by using Green's function, we have

$$u(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \int_{\Omega} G(\mathbf{y},\mathbf{x}) u^2(\mathbf{y}) d\mathbf{y} \ , =: [u](\mathbf{x}) \tag{39}$$

where

$$u^{(0)}(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial G}{\partial n_y}(\mathbf{y},\mathbf{x})\, h(\mathbf{y}) d\mathbf{y} \tag{40}$$

**Definition 6** *Define $\mathcal{S}$ as the space of continuous functions of $\mathbf{x} \in \overline{\Omega}$ equipped with the sup norm.*

**Theorem 7** *If $\|h\|_{\partial\Omega}$ is small enough then* (39) *has a unique solution in a ball of size $2\|h\|_{\partial\Omega}$.*

PROOF. We know that $u^{(0)}$ is the solution of Laplace equation with boundary data $h$, and therefore from max/min principle, it follows that

$$\|u^{(0)}\| \leq \|h\|_{\partial\Omega} \tag{41}$$

Now, we know that

$$\left| \int_{\Omega} \left| G(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right| \right| \leq C \tag{42}$$

independent of $\mathbf{x}$, and therefore, it follows that

$$\|\mathcal{N}[u]\| \leq \|h\|_{\infty} + C\|u\|^2 \leq \|h\|_{\infty} + 4C\|h\|_{\infty}^2 \leq 2\|h\|_{\infty} \tag{43}$$

Similarly, it follows that

$$\|\mathcal{N}[u] - \mathcal{N}[v]\| \leq C\|(u+v)\|\|u-v\| \leq 4C\|h\|_{\infty}\|u-v\| \leq \alpha\|u-v\| \tag{44}$$

Therefore, contractivity is assured and we have a unique solution to the integral equation $u = \mathcal{N}[u]$. □

# 3 General Usage of Contraction Mapping

Consider a nonlinear problem that is written abstractly as

$$\mathcal{N}[u] = 0 \ , \tag{45}$$

for some operator $\mathcal{N}$. Boundary initial data is incorporated in this framework by putting restriction on class of $u$ on which $\mathcal{N}$ is allowed to operate. Suppose we expect a solution $u$ to be near some $u^{(0)}$. In the case of the Burger's equation, we chose $u^{(0)}$ was a solution of the linear heat equation. In the case of nonlinear elliptic problem, we chose this to be the solution to $\Delta u = 0$ with given boundary data. Then, to determine a solution $u$ in the neighborhood fo $u^{(0)}$ we define

$$E = u - u^{(0)} \tag{46}$$

Then, it is clear that $E$ will satisfy

$$\mathcal{N}[u_0 + E] - \mathcal{N}[u_0] = -\mathcal{N}[u_0] =: R \tag{47}$$

Suppose we a linear operator $\mathcal{L}$ defined such that

$$\mathcal{M}[E] := \mathcal{N}[u_0 + E] - \mathcal{N}[u_0] - \mathcal{L}E \tag{48}$$

has the property that in some suitable Banach space $\mathcal{S}$

$$\|\mathcal{M}[E]\| \leq C_0 \|E\|^2 \tag{49}$$

Then, we may write the equivalent problem for $E$ as

$$\mathcal{L}E = R + \mathcal{M}[E] \tag{50}$$

If $\mathcal{L}$ is invertible in $\mathcal{S}$ with

$$\|\mathcal{L}^{-1} G\| \leq C_1 \|G\| \tag{51}$$

then, we obtain

$$E = \mathcal{L}^{-1} R + \mathcal{L}^{-1} \mathcal{M}[E] =: \mathcal{N}_1[E] \tag{52}$$

Now, we notice that from assumption $R$ is small, and therefore, we have the estimate

$$\|\mathcal{N}_1[E]\| \leq C_1 \|R\| + C_1 C_0 \|E\|^2 \tag{53}$$

When $\|R\|$ is sufficiently small, we can show easily under the above assumptions that $\mathcal{N}_1$ maps a ball of size $2C_1\|R\|$ back to itself and is contractive over there. This is a fairly general characterization of the set of all problems that are amenable to contraction mapping methods.