Week 2, Math 6451

1 Linear 1st order PDEs in two independent variables

Consider first a special class of linear 1st order PDEs in two independent variables (x_1, x_2):

$$a_1(x_1, x_2)u_{x_1} + a_2(x_1, x_2)u_{x_2} = c(x_1, x_2) \tag{1}$$

where a_1, a_2 and c are continuous function in some domain $\Omega \subset \mathbb{R}^2$. Denote $\mathbf{x} = (x_1, x_2)$. We will assume the following:

1. On some differentiable curve $\Gamma = \{\mathbf{x} : \mathbf{x} = \mathbf{x}_0(s) \ \ , \ 0 \leq s \leq b\}$, the tangent

$$\frac{d\mathbf{x}_0}{ds} \nparallel \mathbf{a}(\mathbf{x}_0(s)), \tag{2}$$

   where $\mathbf{a} \equiv (a_1, a_2)$. This is called the *non-characteristic* condition, and its significance will be clear later.

2. On the non-characteristic curve Γ, we specify initial condition:

$$u(\mathbf{x}_0(s)) = u_0(s) \tag{3}$$

We now seek a solution in a domain Ω adjoining Γ.

We notice (1) geometrically implies that the directional derivative of u along $\mathbf{a}$ is specified since (1) may be written as

$$\mathbf{a} \cdot \nabla u = c \tag{4}$$

We introduce *characteristic* curves $\mathbf{X}(t; s)$ parametrized by $t \in \mathcal{I} \subset \mathbb{R}$, with $0 \in \mathcal{I}$ for each $0 \leq s \leq b$ such that

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{a}(\mathbf{X}(t; s)) \ , \quad \text{with initial condition } \mathbf{X}(s; 0) = \mathbf{x}_0(s) \tag{5}$$

A unique $\mathbf{C}^1$ solution $\mathbf{X}(t; s)$ for each s is guaranteed locally from theory of ODEs for sufficiently small size of interval $\mathcal{I}$. On such a curve $\mathbf{X}(t, s)$, (4)-(5) imply

$$\frac{\partial}{\partial t} u(\mathbf{X}(t; s)) = c(\mathbf{X}(t; s)) \quad , \quad u\left(\mathbf{X}(0, s)\right) = u_0(s) \tag{6}$$

The theory of ODEs guarantees a locally unique solution to (6) for $t \in \mathcal{I}$, some open interval containing $t = 0$. Denote this solution by

$$u = U(t; s) \tag{7}$$

Note however, this solution process requires *non-characteristic* condition (2), as otherwise, $\frac{d\mathbf{x}_0(s)}{ds} \cdot \nabla u = u_0'(s)$ for arbitrary u_0 will be incompatible with (6) at a point of tangency.

Also, *non-charactertistic* condition (2) above implies that the inverse function theorem applies locally in a neighborhood of $t = 0$, since at $t = 0$,

$$\frac{\partial(X_1, X_2)}{\partial(t, s)} = (a_1, a_2) \cdot (X_{2s}, -X_{1s}) \neq 0 \ . \tag{8}$$

Thus $\mathbf{x} = \mathbf{X}(t; s)$ may be inverted locally for small enough t to obtain

$$(t, s) = (T(\mathbf{x}), S(\mathbf{x})) \tag{9}$$

Thus, using (7), we have solution to the *initial value problem* (consisting of PDE and given initial condition):

$$u(\mathbf{x}) = U(T(\mathbf{x}), S(\mathbf{x})) \tag{10}$$

You may verify this to be true by directly substituting (10) into the original PDE and using ∇T, ∇S that follows from inverse function theorem.

The method of characteristics introduced here is not limited to two independent variables. Indeed, in general, in n independent variables, the initial data is given on a *non-characteristic* $n-1$ dimensional surface, characterized by real parameters $(s_1, s_2, ..s_{n-1}) \equiv \mathbf{s}$ so that vector $\mathbf{a}$ is no where tangent to this surface. Then the procedure above generalizes, if we replace scalar s by vector $\mathbf{s}$.

1.1 Example of an explicit calculation

Consider for instance

$$u_{x_1} + x_2 u_{x_2} = 0 \quad \text{with initial condition } u(0, x_2) = f(x_2) \tag{11}$$

for some differentiable function $f(x_2)$, and the domain of u is all of $\mathbb{R}^2$. It is clear that the curve $\Gamma = \{(x_1, x_2) : x_1 = 0\}$, characterized by parameter $s = x_2$ is everywhere *non-characteristic* since the tangent vector $(0, 1)$ is not parallel of $\mathbf{a} = (1, x_2)$ for *any* value of x_2. Hence, it is proper to specify initial data on Γ. Further, the characteristic starting from each point on Γ is determined by

$$\frac{d}{dt}(X_1(t; s), X_2(t; s)) = (1, X_2(t; s)) \quad \text{with} \quad (X_1(0; s), X_2(0; s) = (0, s) \tag{12}$$

The solution is clearly

$$x_1 = X_1(t; s) \equiv t \quad ; \quad x_2 = X_2(t; s) \equiv s e^t \tag{13}$$

For fixed s, on each characteristic curve $(X_1(t; s), X_2(t; s)$, we have from the PDE:

$$\frac{d}{dt} u = 0 \quad \text{with initial condition } u(X_1(0; s), X_2(0; s)) = f(s) \tag{14}$$

Hence

$$u = U(t; s) = f(s) \tag{15}$$

Inverting (13),

$$t = T(x_1, x_2) = x_1 \quad ; \quad s = S(x_1, x_2) = x_2 e^{-x_1} \tag{16}$$

Therefore,

$$u(x, t) = f(S(x_1, x_2)) = f(x_2 e^{-x_1}) \tag{17}$$

Indeed, we can directly check that (17) solves the PDE for a differentiable f and that it also satisfies the given initial condition on $x_1 = 0$.

2 1st Order quasi-linear PDEs in 2 dimensions

The method of characteristics works for quasi-linear PDEs as well:

$$a_1(x_1, x_2, u)\frac{\partial u}{\partial x_1} + a_2(x_1, x_2, u)\frac{\partial u}{\partial x_2} = c(x_1, x_2, u) \tag{18}$$

with initial condition

$$u = u_0(s) \ \text{ on } \ \Gamma := \left\{ \mathbf{x} \in \mathbb{R}^2,\ \mathbf{x} = \mathbf{x}_0(s), s \in [0, b] \right\} \ , \tag{19}$$

where $\mathbf{x}_0(s), u_0(s) \in \mathbf{C}^1[0, b]$. We will assume $\mathbf{a} = (a_1, a_2)$ and c to be in $\mathbf{C}^1$. Further, we will assume the *non-characteristic* condition that $\mathbf{a}(\mathbf{x}_0(s), u_0(s))$ is *not* parallel to $\mathbf{x}_0'(s)$ for any $s \in [0, b]$. We show the situation geometrically in figure 1. Corresponding to Γ in the (x_1, x_2)

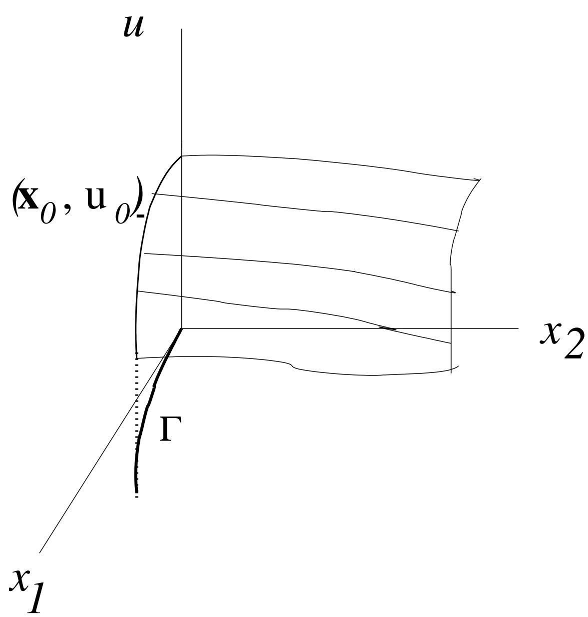

Figure 1: Solution Curves and Solution Surface for quasi-linear PDEs

plane, there is a curve $(x_{1,0}(s), x_{2,0}(s), u_0(s)) \equiv (\mathbf{x}_0(s), u_0(s))$ in the (x_1, x_2, u) space which will be referred to as the initial curve. At each point of this curve, characterised by parameter s, we generate a trajectory by solving the coupled system of two nonlinear ODEs (20) and (21) below:

$$\frac{d\mathbf{x}}{dt} = \mathbf{a}(\mathbf{x}, u) \ \text{ with } \ \mathbf{x}(0; s) = \mathbf{x}_0(s) \tag{20}$$

$$\frac{du}{dt} = c(\mathbf{x}, u) \ \text{ with } \ u(0; s) = u_0(s) \tag{21}$$

For each fixed s, equations (20) and (21) generate solutions to quasi-linear PDE (18) along characteristic curves determined from (20) since (18)-(20) imply

$$c = \frac{d}{dt}u(\mathbf{x}(t)) = \frac{d\mathbf{x}}{dt} \cdot \nabla u(\mathbf{x}(t), t) = \mathbf{a} \cdot \nabla u$$

Since $\mathbf{a}(\mathbf{x}_0(s), u_0(s))$ is no where parallel to $\frac{d\mathbf{x}_0(s)}{ds}$, it follows $\mathbf{x}_t(0; s)$ is not parallel to Γ. We denote the unique $\mathbf{C}^1$ solution to (20) and (21) guaranteed by ODE theory by $(\mathbf{X}(t; s), U(t; s))$. The *solution curve* generated by marching in t for fixed s is shown in Figure 1. Note that the figure only shows t marching in one direction, say with t increasing. Depending on what the intended domain Ω is of the PDE, we may need to march in the direction of decreasing t as well. The union of all such solution curves generates the *solution surface*, sketched in Figure 1. To complete the solution, we need to invert the relation $(x_1, x_2) = \mathbf{X}(t; s)$ to find $(t, s) = (T(x_1, x_2), S(x_1, x_2))$. This is possible locally near Γ since the *non-characteristic* condition on Γ implies that the Jacobian of the transformation $\frac{\partial(X_1, X_2)}{\partial(t,s)}$ is nonzero at $t = 0$ (check this yourself). Thus, the solution of the PDE (18), with initial condition (19) is

$$u = U(T(x_1, x_2), S(x_1, x_2) \tag{22}$$

You can directly verify that this is a solution to the PDE by using the differential equation relations (20) and (21) and noting the relation between derivatives of T, S and derivatives of X_1, X_2 from inverse function theorem.

The method of characteristics is not limited to 2-D. It can be applied to 1st order PDEs with n-variables. In that case, the initial curve Γ is replaced by an $(n-1)$-dimensional initial surface, characterized by vector parameter $\mathbf{s} = (s_1, s_2, ..., s_{n-1})$, were u is specified. The *non-characteristic* condition on Γ is that $\mathbf{a}(\mathbf{x}, u_0(\mathbf{s}))$ is never tangential to Γ.

Also, it is to be noted that for nonlinear 1st order PDEs[1], the solution generally exists only locally in a neighborhood of the initial curve Γ. Smooth solutions for *all* $\mathbf{x}$ is *not* to be expected. This will be illustrated in the example below:

2.1 Example of a simple nonlinear PDE (inviscid Burger's equation)

Consider the nonlinear equation[2]

$$u_{x_2} + u u_{x_1} = 0, \quad \text{with} \quad u(x_1, 0) = f(x_1) \quad \text{with } x_2 > 0 \tag{23}$$

The initial curve is $\Gamma = \{(x_1, x_2) : (x_1, x_2) = (s, 0) = \mathbf{x}_0(s)\}$ Note that $\frac{d}{ds}\mathbf{x}_0(s) = (1, 0)$ is nowhere parallel to $\mathbf{a} = (u, 1)$. Therefore, the non-characteristic condition is met. Now, according to the theory above, we find solution curve for each s by solving coupled system of ODEs

$$\frac{d\mathbf{x}}{dt} = (u, 1) \quad , \quad \frac{du}{dt} = 0 \quad \text{with} \quad \mathbf{x}(0; s) = \mathbf{x}_0(s), \quad u(0; s) = f(s) \tag{24}$$

The solution to the ODEs (24) satisfying the initial conditions is found to be

$$u = U(t; s) = f(s) \tag{25}$$

[1] More generally, the statement is true for the class of hyperbolic PDEs

[2] It is common to use independent variable symbols (x, t) instead of (x_1, x_2), since t has the connotation of . We avoided this so that t is not confused with the characteristic marker variable t

$$\mathbf{x} = \mathbf{X}(t; s) = (s + f(s)t, t) \tag{26}$$

Note that in this case, s is the value of x_1 at $t = 0$, *i.e.* on the initial curve Γ. To complete the solution we need to solve for (t, s) in terms of (x_1, x_2). This is possible in principle for small t since the Jacobian $\frac{\partial(X_1, X_2)}{\partial(t,s)} = -1 \neq 0$. However, explicit expressions are only possible for particularly simple choice of $f(s)$. Otherwise, we think of (26) as providing the inversion $(t, s) = (T(x_1, x_2), S(x_1, x_2))$ in an implicit form, and the solution to PDE will be

$$u = f(S(x_1, x_2)) \tag{27}$$

Besides describing the solution in terms of Fig. 1, it is instructive to look at the characteristic curves in the (x_1, x_2) domain, as in Figures 2 and 3. In the case when $f(s)$ is increasing with s,the initial value of x_1 (see eqn (26)), we note that each point in the (x_1, x_2)-plane is associated with a unique characteristic curve that passes through it corresponding to a unique s. Hence we can invert and find $(t, s) = (T(x_1, x_2), S(x_1, x_2)) = (x_1, S(x_1, x_2)$ for any (x_1, x_2). Classical solution will exist in this case for any (x_1, x_2) for $x_2 > 0$ if $f \in \mathbf{C}^1$. This is really an exceptional case. Generically, $f(s)$ will decrease in some interval in s. Hence the characteristic curves intersect

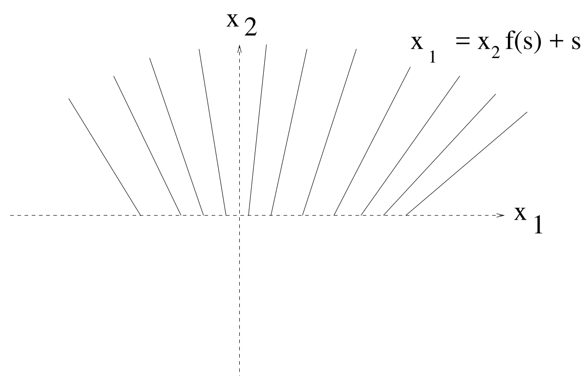

Figure 2: Characteristic Curves for (22) in the (x_1, x_2) plane for increasing f

and cross each other for some set of (x_1, x_2), as shown in Fig. 3 This means that the inversion process to find $s = S(x_1, x_2)$ must fail beyond $x_2 > x_b$ for some x_b shown in Fig. 3.

To see what happens in such cases, we plot $u(x_1, x_2)$ against x_1 for different x_2, as shown in Fig. 4, where we assumed $0.5 < x_b < 3$. Equations (25) implies that a point (x_1, u) curve initially, *i.e.* at $x_2 = t = 0$, is translated to $(x_1 + f(s)x_2, u)$ for $x_2 > 0$, where s is the initial value of x_1, and $f(s)$ is the initial profile of u. Since the translation for points with larger f is larger, it follows that the curve will fold into an backward S-shape and $u(x_1, x_2)$ will become *multi-valued* for $x_2 > x_b$ for some critical x_b that depends on f. The multi-valued solution makes no sense and has to be replaced by a weak solution that allows solution and/or its derivatives to be discontinous. This is a generic feature of nonlinear PDEs.

Later in the course, as time allows, we will discuss how weak solution theory may be developed in nonlinear contexts to accommodate solutions which are not $\mathbf{C}^1$. Below, we illustrate how this is done for a simple linear case.

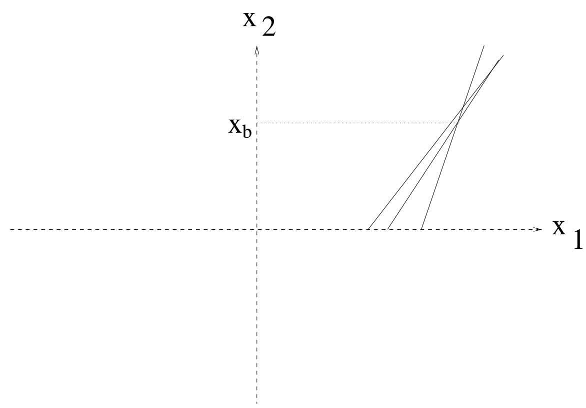

Figure 3: Intersecting Characteristic Curves for (22) for locally decreasing $f(s)$

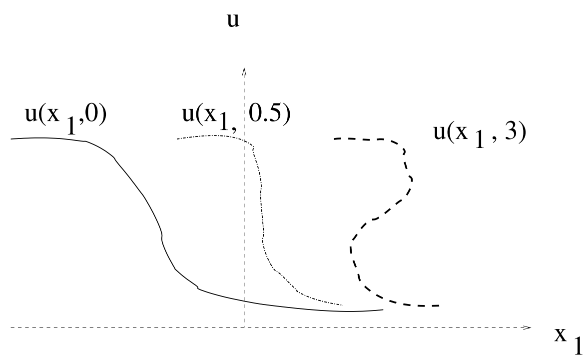

Figure 4: Profile of $u(x_1, x_2)$ as a function of x_1 for different x_2

3 Introduction to Weak Solution Concept

In the last week notes, you will see references to *weak* solution. A *classical solution* to a PDE is one where requires that the solution u has as many derivatives as needed for the equation to make sense. For instance a classical solution to Laplace equation $\Delta u = 0$ must be in $\mathbf{C}^2$. Similarly solution to the heat equation $u_t = \Delta u$ must be in $\mathbf{C}^2$ in $\mathbf{x}$ and $\mathbf{C}^1$ in time t. However, it becomes necessary sometimes *not* to require solution to have as much smoothness. We like to be able to say for instance that $u(x,t) = f(x - ct) + g(x + ct)$ is a solution to the wave equation in one space dimension: $u_{tt} - c^2 u_{xx}$ without requiring $f, g \in \mathbf{C}^2$. For a large class of nonlinear PDEs, this extension of the notion of a solution is necessary because there may not be any classical solution.

In other cases, the introduction of weak solution is a convenient technical tool, because in many cases it is easier to first prove that there there exists *weak* solutions, followed by proof of regularity than to show that directly that a classical solution exists.

We illustrate the concept of solution concept through a simple PDE. Consider

$$u_t + cu_x = 0 \ \text{ for constant } \ c \tag{28}$$

Using method of characteristics, we know this has a classical solution $u(x,t) = f(x - ct)$ for $f \in \mathbf{C}^1$. We seek to relax this assumption on f. We introduce a class of test functions, denoted generically by ϕ, each of which is a a smooth function of (x,t) and has a compact support, *i.e.* vanishes outside a bounded set $\subset \mathbb{R}^2$.

Definition 1 *We define $u(x,t)$ to be a weak solution to (28) if it satisifies*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\phi_t + c\phi_x) u dx dt = 0 \tag{29}$$

for any test function ϕ.

Lemma 2 *If u is a weak solution of (28), but is $\mathbf{C}^1$ in both x and t, then u is a classical (strong) solution to (28). Further, a classical (strong) solution is always a weak solution.*

Proof.

Simply integrate by parts (29) with respect to t or x, depending on whether it is the term ϕ_t or ϕ_x. Noting no contribution at the end points (because of compactness assumption on support of ϕ), it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u_t + cu_x)\phi dx dt = 0 \tag{30}$$

Since $u_t + cu_x$ is continuous in (x,t) and the above equation holds for any ϕ, it follows that u satisfies (28). To show classical solution (strong solution) is also a weak solution, we note that (30) holds for any classical solution. Integrating by parts we obtain (28). $\square$

We now show that for the problem (28), $u(x,t) = f(x - ct)$ is a *weak* solution when $f \in \mathbf{C}^0$ only. To do so, we must show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\phi_t + c\phi_x\right] f(x - ct) dx dt = 0 \tag{31}$$

for any test function ϕ. It is convenient to introduce change of variable $(x,t) \to (x - ct, t) \equiv (\xi, t)$ in the integration in (31). Using chain rule, it is clear that if we write $\phi(x(\xi,t),t) = \Phi(\xi,t)$, then $\Phi_t = \phi_t + c\phi_x$. Therefore, that the left hand side of (31) reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_t f(\xi) dt d\xi \tag{32}$$

This is zero since t integration of Φ_t is zero, as Φ is zero at outside a finite region in the (x,t) plane.

More generally, *weak* solution of

$$a(x,t)u_t + b(x,t)u_x = c(x,t)u \tag{33}$$

may be defined as one that satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(a\phi)_t + (b\phi)_x + c\phi\right] u(x,t) dx dt \tag{34}$$

for any complactly supported test function ϕ.

4 Classification of 2nd order Linear PDEs

Consider a general second order linear PDE in n independent variables:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = g \tag{35}$$

for constants a_{ij}, a_i and a_0. Since the mixed derivative $u_{x_i x_j} = u_{x_j x_i}$, there is no loss of generality in taking $a_{ij} = a_{ji}$. Consider a linear change in variable (for constant $n \times n$ materix B):

$$\xi = (\xi_1, \xi_2, ..\xi_n)^T = B\mathbf{x} \tag{36}$$

Then, it follows on using chain rule that

$$\sum_{i,j} a_{ij} u_{x_i x_j} = \sum_{k,l} \left(\sum_{i,j} b_{ki} a_{ij} b_{lj} \right) u_{\xi_k \xi_l} \quad , \quad \sum_i a_i u_{x_i} = \sum_k \left[\sum_i b_{ki} a_i \right] u_{\xi_k} \tag{37}$$

Hence the PDE (35) reduces to

$$\sum_{k,l} c_{kl} u_{\xi_k \xi_l} + \sum_k c_k u_{\xi_k} + a_0 u = g \tag{38}$$

Therefore, the coefficient matrix of the PDE has been changed from A (defined by elements a_{ij}) to $C = BAB^T$ (C is defined here by elements c_{kl}). Since we arranged A to be a symmetric real matrix, there exists an orthogonal matrix B with determinant 1 (i.e. rotation matrix) so that

$$B^T AB = D = \begin{pmatrix} d_1 & 0 & .. \\ 0 & d_2 & .. \\ . & . & . \end{pmatrix} \tag{39}$$

a diagonal matrix. With this choice of B, the PDE (38) has the form

$$d_1 u_{\xi_1 \xi_1} + d_2 u_{\xi_2 \xi_2} \; + \; ..d_n u_{\xi_n \xi_n} + \sum_k c_k u_{\xi_k} + a_0 u = g \tag{40}$$

Further note that by simply rescaling ξ_j by factors of $\sqrt{|d_j|}$, we can ensure that the coefficient of the second derivative terms are all either +1 or -1 for all terms for which $d_j \neq 0$.

Definition 3 *The PDE (35) is called elliptic if all eigenvalues d_1, d_2, $..d_n$ are positive or all are negative (This is equivalent to matrix A (or $-A$) being positive definite). The PDE is called hyperbolic if none of d_1, d_2, $..d_n$ vanish and one of them has the opposite sign of the $(n-1)$ others. If exactly one eigen values is zero and all the others have the same sign, the PDE is called parabolic.*

Remark 1 *When $n = 2$, the condition for being elliptic, hyperbolic and parabolic can be shown to reduce to $a_{12}^2 < a_{11}a_{22}$, $a_{12}^2 > a_{11}a_{22}$ and $a_{12}^2 = a_{11}a_{22}$, respectively. Also, note that for $n > 2$, there can be equations that do not belong to any of these three categories. For instance, if no eigenvalues vanish and at least two of them are positive and at least another two negative, then it is referred to as ultra-hyperbolic.*

You can directly check that Laplace equation $\Delta u = 0$ is *elliptic*, wave equation $u_{tt} - \Delta u = 0$ is *hyperbolic* and heat equation $u_t - \kappa \Delta u = 0$ is *parabolic*. It also follows from (40) that there exists change of variable independent variables so that any elliptic equation of the form (35) can be transformed into

$$\Delta u + ..(\text{lower order derivatives of } u) = g \tag{41}$$

while hyperbolic equations can be transformed to

$$u_{tt} - \Delta_{(n-1)} u + ..(\text{lower order derivatives}) = g \tag{42}$$

where we choose the $t = \xi_l$ (after rescaling by $\sqrt{|d_l|}$), l being the index for which d_l has opposite sign from other eigen values of A, and where $\Delta_{(n-1)}$ is is the Laplacian operator in $n-1$ variables $(\xi_1, \xi_2, , , \xi_{l-1}, \xi_{l+2}, ...\xi_n)$, after rescaling by factors of $\sqrt{|d_k|}$, as mentioned before. It can also be shown that *parabolic* equations can be transformed to

$$\Delta_{(n-1)} u + ..(\text{lower order derivatives}) = g \tag{43}$$

This is the reason why study of Laplace, Wave Equation and Diffusion (or Heat) gives general idea of the behavior of Elliptic, Hyperbolic and Parabolic class of differential equations.

5 2nd Order Linear Wave Equation in 1-D

We consider for constant c,

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad -\infty < x < \infty \tag{44}$$

We notice that we can factorize operator $\partial_t^2 - c^2 \partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$. Hence, if we introduce a new variable $v = u_t + cu_x$, then it follows that v must satisfy $v_t - cv_x = 0$. From using the method of characteristics, we know $v(x,t) = h(x+ct)$. Therefore, (44) reduces to

$$u_t + cu_x = h(x+ct)$$

We now note that a particular solution to the above is $u(x,t) = f(x+ct)$ where $f'(s) = \frac{h(s)}{2c}$. So, we can decompose $u(x,t) = f(x+ct) + w(x,t)$ and obtain

$$w_t + cw_x = 0$$

From method of characteristics again, we know $w(x,t) = g(x-ct)$. Hence the general solution to (44) is given by

$$u(x,t) = f(x+ct) + g(x-ct) \tag{45}$$

which is a classical solution for $f,\, g \in \mathbf{C}^2$.

Alternate method of finding the solution is to introduce change of variables $(\xi, \eta) = (x+ct, x-ct)$. Then, on using chain rule, it can be easily shown that PDE (44) reduces to $-4c^2 u_{\xi\eta} = 0$, which on integration with respect to ξ and η gives rise to (45).

We now discuss solution to (44) satisfying initial conditions

$$u(x,0) = \phi(x) \quad ; \quad u_t(x,0) = \psi(x) \tag{46}$$

If we use the representation (45), it follows that

$$u(x,0) = f(x) + g(x) = \phi(x) \tag{47}$$

and

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x) \tag{48}$$

Hence $f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi(s')ds' + A$ and $g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(s')ds' + B$ for constants A and B. But since $\phi(s) = f(s) + g(s)$, $A + B = 0$. It follows that

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{49}$$

This is called the *D'Alembert* solution. One can directly check that (49) is a classical solution to the PDE and satisfies the initial conditions provided $\phi \in \mathbf{C}^2$ and $\psi \in \mathbf{C}^2$.

5.1 Domains of Dependence and Influence

It is to be noted from expression (49) that the solution at any point (x,t) only depends on initial conditions $\phi(\xi)$ and $\psi(\xi)$ for $\xi \in (x-ct, x+ct)$. Since the equations are autonomous, (49) also can be used to relate solution $u(x,t)$ to solution at time $t-\tau$:

$$u(x,t) = \frac{1}{2}\left(u(x+c\tau, t-\tau) + u(x-c\tau, t-\tau)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(s, t-\tau)ds \tag{50}$$

where u and u_t at time $t-\tau$ are calculated using (49). Indeed, in the $x-t$ plane, if we a triangle with vertices at (x,t), $(x-ct, 0)$ and $(x+ct, 0)$; the solution at the vertex (x,t) is only influenced by the solution at earlier t within this triangle. This triangle is referred to as the *domain of dependence*. Anything outside this triangle has no influence on the solution at (x,t).

Similarly, if we ask the question what is the domain later in time which is influenced by the solution at a point (x,t) influence. Equation (49) implies that a point (x,t) only influences solution inside the cone with slopes $\pm c$ with vertex at (x,t). This is the *domain of influence*. This is referred to as the principle of causality in the text.

This is a generic feature of *hyperbolic* partial differential equation–solution at any point is only by parts of the initial and boundary conditions; also solution at any (x,t) affects solution later in time, if it is inside the domain of influence. Physically, consider propagation of electromagnetic radiation, say light, emanating from some source at $t=0$. Note this propagation is governed by wave equation. Clearly this light will not felt at time t at a point $\mathbf{x}$, if there was not enough time for the light to have travelled the distance from the source.

5.2 Energy and Uniqueness of Solution of IVP

We go back to the idealization of a homogeneous vibrating string for $x \in (-\infty, \infty)$. The Kinetic Energy (KE) of this string will be $KE = \frac{\rho}{2}\int_{-\infty}^{\infty} u_t^2 dx$. It's rate of change is

$$\frac{d}{dt}KE = \rho \int_{-\infty}^{\infty} u_t u_{tt} dx$$

Using the PDE for the vibrating string: $\rho u_{tt} = T u_{xx}$ we obtain

$$\frac{d}{dt}KE = T\int_{-\infty}^{\infty} u_t u_{xx} dt = -T\int_{-\infty}^{\infty} u_{tx} u_x dt = -\frac{d}{dt}\frac{T}{2}\int_{-\infty}^{\infty} u_x^2$$

Therefore, the expression for string potential energy (PE) due to tension must be

$$PE = \frac{T}{2}\int_{-\infty}^{\infty} u_x^2,$$

so that the equation before reads as conservation of total energy $E = PE + KE$

$$\frac{d}{dt}E = 0,$$

where

$$E = \int_{-\infty}^{\infty}\left(\frac{\rho}{2}u_t^2 + \frac{T}{2}u_x^2\right) dx$$

In general for wave equation in the form

$$u_{tt} = c^2 u_{xx},$$

we have conservation of 'energy' E from the equations, if we define it as

$$E = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) dx$$

The conservation of energy provides an easy proof of uniqueness:

Lemma 4 *For the linear wave equation (44), with specified initial condition (46), any solution that is a priori* $\mathbf{C}^1$*, is unique.*

PROOF.

Assume two solutions u and v satisfying both (44) and (46) with the same ϕ and ψ. It follows that $w = u - v \in \mathbf{C}^1$ and satisfies (44) and zero initial conditions. Since Energy is conserved in time, it follows $E = 0$, since $w = 0$ and $w_t = 0$ initially. Hence it follows $w_x = 0$ and $w_t = 0$ for all times. So, $w =$ const., independent of t. Since $w = 0$ initially, $w = 0$ for all t.

□