1 Diffusion in $\mathbb{R}^n$

Recall that for scalar $x$,

$$S(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp \left[ -\frac{x^2}{4\kappa t} \right]$$  \hspace{1cm} (1)

is a special solution to 1-D heat equation with properties

$$\int_{\mathbb{R}} S(x, t) dx = 1 \text{ for } t > 0, \text{ and yet } \lim_{t \to 0^+} S(x, t) = 0 \text{ for fixed } x \neq 0$$  \hspace{1cm} (2)

This was called a source solution of heat equation with source at the origin.

We now claim that the product $S(x, t) \equiv S(x_1, t)S(x_2, t)S(x_3, t)...S(x_n, t)$ is a solution to the heat equation in $\mathbb{R}^n$:

$$u_t = \kappa \Delta$$  \hspace{1cm} (3)

and satisfies property

$$\int_{\mathbb{R}^n} S(x, t) dx = 1 \text{ for } t > 0, \text{ and yet } \lim_{t \to 0^+} S(x, t) = 0 \text{ for fixed } x \neq 0$$  \hspace{1cm} (4)

We note that by using product rule

$$\frac{\partial S}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial t} S(x_j, t) \prod_{i \neq j} S(x_i, t) = \kappa \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} S(x_j, t) \prod_{i \neq j} S(x_i, t) = \kappa \Delta S$$  \hspace{1cm} (5)

Further,

$$\int_{\mathbb{R}^n} S(x, t) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} ... \int_{\mathbb{R}} S(x_1, t)S(x_2, t)...S(x_n, t) dx_1 dx_2 ... dx_n = 1$$  \hspace{1cm} (6)

Further if $x \neq 0$, then direct examination of

$$S(x, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp \left[ -\frac{x_1^2 + x_2^2 + ... + x_n^2}{4\kappa t} \right]$$  \hspace{1cm} (7)

shows that $\lim_{t \to 0^+} S(x, t) = 0$ for fixed $x \neq 0$. The solution $S$ is the source solution in $\mathbb{R}^n$.

Analogous to 1-D, we have the following theorem:

**Theorem 1** The solution to the heat equation in $\mathbb{R}^n$ that satisfies initial condition

$$u(x, 0) = \phi(x) \text{ for } \phi \in C^0(\mathbb{R}^n)$$  \hspace{1cm} (8)

is given by

$$u(x, t) = \int_{\mathbb{R}^n} S(x - y, t)\phi(y)dy$$  \hspace{1cm} (9)
2 Diffusion in the half-line

2.1 Dirichlet Boundary condition

We consider solution to heat equation in 1-D, with \( x \in \mathbb{R}^+ \) and take the Dirichlet boundary condition at \( x = 0 \). So the problem is

\[
v_t - \kappa v_{xx} = 0 \quad \text{for} \quad x > 0, \quad t > 0
\]
\[
v(x, 0) = \phi(x)
\]
\[
v(0, t) = 0
\]

We seek to find a solution to this problem explicitly. If it exists, the classical solution for which \( v(x, t) \to 0 \) as \( x \to \infty \) is unique by applying maximum principle or energy method.

Now the initial data \( \phi(x) \) is only specified for \( x > 0 \).

We do an odd extension, i.e. define an extended function \( \phi_{\text{odd}}(x) \) in \( \mathbb{R} \) (see Fig. 1) so that

\[
\phi_{\text{odd}}(x) = \phi(x) \quad \text{for} \quad x > 0 ; \quad \phi_{\text{odd}}(x) = -\phi(-x) \quad \text{for} \quad x < 0
\]

![Odd Extension of \( \phi(x) \) to \( x \in \mathbb{R} \)](image)

We do an odd extension, i.e. define an extended function \( \phi_{\text{odd}}(x) \) in \( \mathbb{R} \) (see Fig. 1) so that

\[
\phi_{\text{odd}}(x) = \phi(x) \quad \text{for} \quad x > 0 ; \quad \phi_{\text{odd}}(x) = -\phi(-x) \quad \text{for} \quad x < 0
\]

Let \( u(x, t) \) be a solution to heat equation so as to satisfy

\[
u(x, 0) = \phi_{\text{odd}}(x) \quad \text{for} \quad x \in \mathbb{R}
\]

Then, applying equation (15) of week 3 notes,

\[
u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi_{\text{odd}}(y)dy
\]
Breaking up the integral into two parts $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ and changing variables $y \to -y$ in the first and using (14), we note that (15) implies that
\[ u(x, t) = \int_{0}^{\infty} [S(x - y, t) - S(x + y, t)] \phi(y) dy \] (16)
We note that this solution automatically satisfies Dirichlet boundary condition $u(0, t) = 0$, and therefore from uniqueness, is the desired solution $v(x, t)$.

Thus, using expressions for $S(x, t)$ from last week notes, we obtain
\[ v(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{0}^{\infty} \left\{ \exp \left[ -\frac{(x - y)^2}{4\kappa t} \right] - \exp \left[ -\frac{(x + y)^2}{4\kappa t} \right] \right\} \phi(y) dy \] (17)
We have just illustrated the method of obtaining solution through odd-extension or reflection about the origin; this is applicable to many other half-line problems involving Dirichlet boundary conditions.

### 2.2 Neumann Boundary Condition and even extension

We now consider the diffusion problem on a half-line but with Neumann condition. The problem becomes
\[ w_t - \kappa w_{xx} = 0 \quad \text{for} \quad x > 0, \quad t > 0 \] (18)
\[ w(x, 0) = \phi(x) \] (19)
\[ w_x(0, t) = 0 \] (20)
In this case, it is more convenient to find solution through an even extension. We define $\phi_{\text{even}}(x)$
\[ \phi_{\text{even}}(x) = \phi(x) \quad \text{for} \quad x > 0; \quad \phi_{\text{even}}(x) = \phi(-x) \quad \text{for} \quad x < 0, \] (21)
and solve the initial value problem
\[ u_t = \kappa u_{xx}; \quad u(x, 0) = \phi_{\text{even}}(x) \] (22)
The solution to this is
\[ u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{even}}(y) dy \] (23)
Once again breaking up the above integral into $\int_{-\infty}^{0} + \int_{0}^{\infty}$ and using change of variable $y \to -y$ in the first integral, and using relation (21), one finds
\[ u(x, t) = \int_{0}^{\infty} [S(x - y, t) + S(x + y, t)] \phi(y) dy \] (24)
On differentiating (24) with respect to $x$ and noting that $S_x(-y, t) = -S_x(y, t)$, it follows that $u_x(0, t) = 0$ for all $t > 0$. Thus the solution (24) indeed solves the Neumann problem for $w$. Using energy method again, we can prove that the classical solution to the initial value problem (18)-(20) is unique. Hence, using expressions for $S(x, t)$, we obtain from (24) solution to (18)-(20) in the form:
\[ w(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{0}^{\infty} \left\{ \exp \left[ -\frac{(x - y)^2}{4\kappa t} \right] + \exp \left[ -\frac{(x + y)^2}{4\kappa t} \right] \right\} \phi(y) dy \] (25)
3 Half-line problem for linear wave equation

Now, we try the same type of reflection approach for second order wave equation. Consider first Dirichlet boundary condition. Thus the problem (IVP) is
\[
\begin{align*}
    v_{tt} - c^2 v_{xx} &= 0 & \text{for } x > 0, -\infty < t < \infty \\
    v(x,0) &= \phi(x), \quad v_t(x,0) = \psi(x) & \text{for } x > 0 \\
    v(0,t) &= 0
\end{align*}
\] (26)
where \( \phi \in C^2, \psi \in C^1 \).

As for the diffusion equation on a line, we carry out an odd extension over \( \mathbb{R} \), in this case both for \( \phi(x) \) and \( \psi(x) \). We define the oddly extended functions to be \( \phi_{\text{odd}}(x) \) and \( \psi_{\text{odd}}(x) \). We seek solution \( u(x,t) \) to wave equation for \( x \in \mathbb{R} \) so that it satisfies
\[
    u(x,0) = \phi_{\text{odd}}(x), \quad u_t(x,0) = \psi_{\text{odd}}(x) \quad \text{for } x \in \mathbb{R}
\] (29)
From d’Alembert formula, the solution is
\[
u(x,t) = \frac{1}{2} \left[ \phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy \]
(30)
We verify that \( u(0,t) = 0 \) is indeed satisfied by this expression. Also, using energy arguments, we can prove uniqueness of solution satisfying (26)-(28). Hence, desired \( u(x,t) \) is given by (30).

Now the formula (30) can be re-expressed in terms of \( \phi(x) \) and \( \psi(x) \), but the expression is different in different regimes of \( (x,t) \).

First, for \( x > c|t| \), we notice that each of the arguments \( x - ct \) and \( x + ct \) are positive. Hence, (30) reduces to
\[
    v(x,t) = \frac{1}{2} \left[ \phi(x+ct) + \phi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)dy
\] (31)
The second regime is \( 0 < x < ct \). We have for \( \phi_{\text{odd}}(x - ct) = -\phi(ct - x) \), and \( \psi_{\text{odd}}(y) = -\psi(-y) \) for \( y < 0 \). Hence
\[
    v(x,t) = \frac{1}{2} \left[ \phi(x+ct) - \phi(ct-x) \right] + \frac{1}{2c} \left[ \int_{0}^{x+ct} \psi(y)dy + \int_{x-ct}^{0} [-\psi(-y)]dy \right]
\]
(32)
Graphically, the result (32) can be interpreted as follows. Draw the backward characteristics from the point \((x,t)\) (see Fig.2). In the regime \( 0 < x < ct \), such a backward characteristic intersects the \( t \)-axis, before crossing the \( x \)-axis at \((x-ct,0)\). The formula (32) shows that the reflection induces a change of sign. The value of \( v(x,t) \) now depends on the values of \( \phi \) at the pair of points \( ct \pm x \) and on \( \psi \) over the interval \((ct-x,ct+x)\), which is shorter than \((x-ct,x+ct)\).

This is because the integral of \( \psi_{\text{odd}}(y) \) over the symmetric interval \((x-ct,ct-x)\) is zero.

If \( 0 < x < -ct \), using similar arguments, we can check that (30) reduces to
\[
    v(x,t) = \frac{1}{2} \left[ -\phi(-ct-x) + \phi(-ct+x) \right] - \frac{1}{2c} \left[ \int_{-ct-x}^{-ct+x} \psi(y)dy \right]
\] (33)
The case of Neumann problem on a half-line for the wave equation is very similar, with an even extension of each of \( \phi \) and \( \psi \).
Consider the 1-D wave equation on a finite interval in $x$ with homogeneous Dirichlet boundary conditions, that would correspond to say a guitar string with fixed ends. The initial value problem is given by:

$$v_{tt} = c^2 v_{xx}, \quad v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x) \quad \text{for} \quad 0 < x < l$$

(34)

and

$$v(0,t) = v(l,t) = 0$$

(35)

We can extend the data for each of $\phi(x)$ and $\psi(x)$ by first doing an odd extension to get a function over $(-l,l)$ and then define periodical extensions $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ over $\mathbb{R}$. Thus

$$\phi_{\text{ext}}(x) = \phi(x) \quad \text{for} \quad x \in (0,l) \quad , \quad \phi_{\text{ext}}(x) = -\phi(-x) \quad \text{for} \quad x \in (-l,0) \quad \text{and} \quad \phi_{\text{ext}}(x+2l) = \phi_{\text{ext}}(x)$$

(36)

and a similar formula is valid for $\psi_{\text{ext}}(x)$. It is then possible to write solution as

$$v(x,t) = \frac{1}{2} [\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(s) ds$$

(37)

It can be checked directly that this satisfies $v(0,t) = 0$. To check $v(l,t) = 0$, we note that $\phi_{\text{ext}}(l - ct) = \phi_{\text{ext}}(-l - ct) = -\phi_{\text{ext}}(l + ct)$. Again, using periodicity and oddness of $\psi_{\text{ext}}$, we can prove that $\int_{l-ct}^{l+ct} \psi(y) dy = 0$.

While the expression (37) for the solution is relatively simple in terms of $\phi_{\text{ext}}$ and $\psi_{\text{ext}}$, it is much more complicated if we want to write it in terms of $\phi$ and $\psi$. There are different regimes of expression, depending on how many times reflection was possible at the end points $x = 0$ and $x = l$. I will refer you to Fig. 4 of the text on page 62 for a graphical illustration.

We will find later in this course alternate expressions of solution in terms of a Fourier Series.
5 Diffusion with a source

In this section, we solve the inhomogeneous diffusion equation in $\mathbb{R}^n$, for any dimension $n \geq 1$. So, the initial value problem of interest is

$$u_t - \kappa \Delta u = f(x, t)$$ (38)
$$u(x, 0) = \phi(x)$$ (39)

where $\phi, f \in C^0$. We will prove that the solution of the problem (38)-(39) is given by

$$u(x, t) = \int_{\mathbb{R}^n} S(x - y, t) \phi(y) dy + \int_0^t \int_{\mathbb{R}^n} S(x - y, t - \tau) f(y, \tau) dy d\tau$$ (40)

This is an example of application of so-called Duhammel’s principle where solution to inhomogeneous linear autonomous differential equation is expressed in terms of appropriate solution to the homogeneous solution, which in this case is $S$. See text page 65 for analogy to ODEs.

It is convenient to define

$$v(x, t) = \int_0^t \int_{\mathbb{R}^n} S(x - y, t - \tau) f(y, \tau) dy d\tau$$ (41)

Then noting

$$\lim_{\tau \to t^-} \int_{\mathbb{R}^n} S(x - y, t - \tau) f(y, \tau) dy = f(x, t)$$

we obtain

$$v_t - \kappa \Delta v = \lim_{\tau \to t^-} \int_{\mathbb{R}^n} S(x - y, t - \tau) f(y, \tau) dy + \int_0^t \int_{\mathbb{R}^n} [S_t(x - y, t - \tau) - \kappa \Delta x S(x - y, t - \tau)] = f(x, t)$$ (42)

Further, since

$$\left| \int_{\mathbb{R}^n} S(x - y, t - \tau) f(y, \tau) dy \right| \leq \sup_{y \in \mathbb{R}^n} |f(y, \tau)| \int_{\mathbb{R}^n} S(x - y, t - \tau) dy = \|f(., \tau)\| \text{ for } \tau < t$$

it follows $\lim_{t \to 0^+} v(x, t) = 0$. Since, we know from before that

$$w(x, t) = \int_{\mathbb{R}^n} S(x - y, t) \phi(y) dy$$

solves the heat equation without any source $w_t - \kappa \Delta w = 0$ and initial condition $w(x, 0) = \phi(x)$, it follows that $u(x, t) = v(x, t) + w(x, t)$ given by (40) solves the inhomogeneous heat equation (38) and satisfies the initial condition (39).

6 Inhomogenous Heat Equation in a half space $\mathbb{H}$

Note that the text on page 67 talks about solving inhomogenous diffusion equation on a half-line. There is no problem extending this idea to $n$-dimensions. We define half space

$$\mathbb{H} \equiv \{x \in \mathbb{R}^n : x_1 > 0\}$$ (43)
Our problem is to solve
\[ v_t - \kappa \Delta v = f(x, t) \quad \text{for} \ x \in \mathbb{H} \quad (44) \]
\[ v(x, 0) = \phi(x) \quad \text{for} \ x \in \mathbb{H} \quad (45) \]
\[ v(x, t) = 0 \quad \text{for} \ x \in \partial \mathbb{H} \quad (46) \]

First, we note using odd-extension of \( \phi(x) \) in the component \( x_1 \), as in 1-D, we can obtain a source type solution that satisfies the boundary condition on \( x_1 = 0 \) (which is the same as \( \partial \mathbb{H} \)). It is convenient to define
\[ y_- = (-y_1, y_2, y_3, \ldots, y_n)_+ \quad \text{where} \ y = (y_1, y_2, \ldots, y_n) \]
Then, it is easy to verify that
\[ T(x, y, t) \equiv S(x - y, t) - S(x - y_-, t) \quad (47) \]
is a solution of the heat equation in the half-place \( \mathbb{H} \) corresponding to a unit source at \( y \) that satisfies the homogenous Dirichlet condition (36). Therefore, using the same ideas as in the last section, we can verify that the solution to the problem posed in (44)-(46) is given by
\[ v(x, t) = \int_{\mathbb{R}^n} T(x, y, t) \phi(y) \, dy + \int_0^t \int_{\mathbb{R}^n} T(x, y, t - \tau) f(y, \tau) \, dy \, d\tau \quad (48) \]

### 7 Note on Inhomogenous Dirichlet or Neumann conditions

So far, our boundary conditions, either for the heat or wave equation involved homogeneous \textit{Dirichlet} or \textit{Neumann} boundary conditions. However, solution for an inhomogenous condition is not a serious problem. For the sake of being definite, consider for instance
\[ u_t - \kappa u_{xx} = f(x, t); \quad \text{for} \ x > 0 \ , \ u(0, t) = h(t) \ , \ u(x, 0) = \phi(x) \quad (49) \]
We assume \( h \in C^1 \). Noting that the function \( w(x, t) \equiv e^{-x} h(t) \) satisfies the boundary condition at \( x = 0 \) and well-behaved in \( x \) as \( x \to +\infty \), it follows that by decomposing \( u(x, t) = w(x, t) + v(x, t) \), \( v(x, t) \) satisfies
\[ v_t - \kappa v_{xx} = f(x, t) - w_t + \kappa w_{xx} \equiv \tilde{f}(x, t) \quad (50) \]
\[ v(0, t) = 0 \ ; \ v(x, 0) = \phi(x) - h(0) e^{-x} \equiv \tilde{\phi}(x) \quad (51) \]
Equations (38)-(39) defines an inhomogeneous heat equation with homogeneous Dirichlet condition and is of the type for which we have a general representation of solution, as seen in the last section.