Week 4 Lectures, Math 6451, Tanveer

1 Diffusion in $\mathbb{R}^n$

Recall that for scalar x,

$$S(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{x^2}{4\kappa t}\right] \tag{1}$$

is a special solution to 1-D heat equation with properties

$$\int_{\mathbb{R}} S(x,t)dx = 1 \quad \text{for} \quad t > 0, \quad \text{and yet} \quad \lim_{t\to 0^+} S(x,t) = 0 \quad \text{for fixed } x \neq 0 \tag{2}$$

This was called a source solution of heat equation with source at the origin.

We now claim that the product $\mathcal{S}(\mathbf{x},t) \equiv S(x_1,t)S(x_2,t)S(x_3,t)...S(x_n,t)$ is a solution to the heat equation in $\mathbb{R}^n$:

$$u_t = \kappa\Delta \quad \text{for } \mathbf{x} \in \mathbb{R}^n \tag{3}$$

and satisfies property

$$\int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x},t)d\mathbf{x} = 1 \quad \text{for} \quad t > 0, \quad \text{and yet} \quad \lim_{t\to 0^+} \mathbf{S}(\mathbf{x},t) = 0 \quad \text{for fixed } \mathbf{x} \neq \mathbf{0} \tag{4}$$

We note that by using product rule

$$\frac{\partial \mathcal{S}}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial t} S(x_j,t) \prod_{i\neq j} S(x_i,t) = \kappa \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} S(x_i,t) \prod_{i\neq j} S(x_i,t) = \kappa\Delta\mathcal{S} \tag{5}$$

Further,

$$\int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x},t)d\mathbf{x} = \int_{\mathbb{R}}\int_{\mathbb{R}}...\int_{\mathbb{R}} S(x_1,t)S(x_2,t)...S(x_n,t)dx_1dx_2...dx_n = 1 \tag{6}$$

Further if $\mathbf{x} \neq 0$, then direct examination of

$$\mathbf{S}(\mathbf{x},t) = \frac{1}{(4\kappa\pi t)^{n/2}} \exp\left[-\frac{x_1^2 + x_2^2 \ + \ ..x_n^2}{4\kappa t}\right] = \frac{1}{(4\kappa\pi t)^{n/2}} \exp\left[-\frac{\mathbf{x}^2}{4\kappa t}\right] \tag{7}$$

shows that $\lim_{t\to 0^+} \mathcal{S}(\mathbf{x},t) = 0$ for fixed $\mathbf{x} \neq 0$. The solution $\mathcal{S}$ is the source solution in $\mathbb{R}^n$. Analogous to 1-D, we have the following theorem:

Theorem 1 *The solution to the heat equation in $\mathbb{R}^n$ that satisfies initial condition*

$$u(\mathbf{x},0) = \phi(\mathbf{x}) \quad \text{for } \phi \in \mathbf{C}^0(\mathbb{R}^n) \tag{8}$$

is given by

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x}-\mathbf{y},t)\phi(\mathbf{y})d\mathbf{y} \tag{9}$$

2 Diffusion in the half-line

2.1 Dirichlet Boundary condition

We consider solution to heat equation in 1-D, with $x \in \mathbf{R}^+$ and take the *Dirichlet* boundary condition at $x = 0$. So the problem is

$$v_t - \kappa v_{xx} = 0 \quad \text{for} \quad x > 0, \quad t > 0 \tag{10}$$

$$v(x,0) = \phi(x) \tag{11}$$

$$v(0,t) = 0 \tag{12}$$

We seek to find a solution to this problem explicitly. If it exists, the classical solution for which $v(x,t) \to 0$ as $x \to \infty$ is unique by applying maximum principle or energy method.

Now the initial data $\phi(x)$ is only specified for $x > 0$.

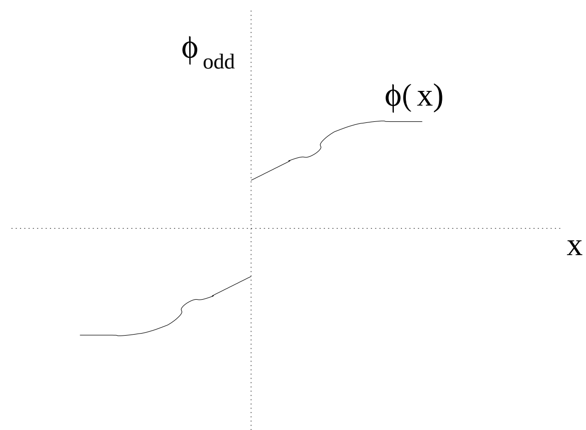

Figure 1: Odd Extension of $\phi(x)$ to $x \in \mathbb{R}$

We do an *odd extension*, *i.e.* define an extended function $\phi_{odd}(x)$ in $\mathbb{R}$ (see Fig. 1) so that

$$\phi_{odd}(x) = \phi(x) \quad \text{for} \quad x > 0 \quad ; \quad \phi_{odd}(x) = -\phi(-x) \quad \text{for} \quad x < 0 \tag{13}$$

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Let $u(x,t)$ be a solution to heat equation so as to satisfy

$$u(x,0) = \phi_{odd}(x) \quad \text{for} \quad x \in \mathbb{R} \tag{14}$$

Then, applying equation (15) of week 3 notes,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)dy \tag{15}$$

Breaking up the integral into two parts $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ and changing variables $y \to -y$ in the first and using (14), we note that (15) implies that

$$u(x,t) = \int_0^\infty \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy \tag{16}$$

We note that this solution automatically satisfies *Dirichlet* boundary condition $u(0,t) = 0$, and therefore from uniqueness, is the desired solution $v(x,t)$ Therefore, using expresssions for $S(x,t)$ from last week notes, we obtain

$$v(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_0^\infty \left\{\exp\left[-\frac{(x-y)^2}{4\kappa t}\right] - \exp\left[-\frac{(x+y)^2}{4\kappa t}\right]\right\}\phi(y)dy \tag{17}$$

We have just illustrated the method of obtaining solution through *odd-extension* or reflection about the origin; this is applicable to many other half-line problems involving *Dirichlet* boundary conditions.

2.2 Neumann Boundary Condition and even extension

We now consider the diffusion problem on a half-line but with *Neumann* condition. The problem becomes

$$w_t - \kappa w_{xx} = 0 \quad \text{for} \quad x > 0, \quad t > 0 \tag{18}$$

$$w(x,0) = \phi(x) \tag{19}$$

$$w_x(0,t) = 0 \tag{20}$$

In this case, it is more convenient to find solution through an *even extension*. We define $\phi_{even}(x)$

$$\phi_{even}(x) = \phi(x) \quad \text{for} \quad x > 0 \quad ; \quad \phi_{even}(x) = \phi(-x) \quad \text{for} \quad x < 0, \tag{21}$$

and solve the initial value problem

$$u_t = \kappa u_{xx} \quad ; \quad u(x,0) = \phi_{even}(x) \tag{22}$$

The solution to this is

$$u(x,t) = \int_{-\infty}^\infty S(x-y,t)\phi_{even}(y)dy \tag{23}$$

Once again breaking up the above integral into $\int_{-\infty}^{0} + \int_{0}^{\infty}$ and using change of variable $y \to -y$ in the first integral, and using relation (21), one finds

$$u(x,t) = \int_0^\infty \left[S(x-y,t) + S(x+y,t)\right]\phi(y)dy \tag{24}$$

On differentiating (24) with respect to x and noting that $S_x(-y,t) = -S_x(y,t)$, it follows that $u_x(0,t) = 0$ for all $t > 0$. Thus the solution (24) indeed solves the Neumann problem for w. Using energy method again, we can prove that the classical solution to the initial value problem (18)-(20) is unique. Hence, using expressions for $S(x,t)$, we obtain from (24) solution to (18)-(20) in the form:

$$w(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_0^\infty \left\{\exp\left[-\frac{(x-y)^2}{4\kappa t}\right] + \exp\left[-\frac{(x+y)^2}{4\kappa t}\right]\right\}\phi(y)dy \tag{25}$$

3 Half-line problem for linear wave equation

Now, we try the same type of reflection approach for second order wave equation. Consider first *Dirichlet* boundary condition. Thus the problem (IVP) is

$$v_{tt} - c^2 v_{xx} = 0 \quad \text{for} \quad x > 0 \quad , -\infty \ < \ t < \ \infty \tag{26}$$

$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x) \quad \text{for} \quad x > 0 \tag{27}$$

$$v(0, t) = 0 \tag{28}$$

where $\phi \in \mathbf{C}^2$, $\psi \in \mathbf{C}^1$.

As for the diffusion equation on a line, we carry out an odd extension over $\mathbb{R}$, in this case both for $\phi(x)$ and $\psi(x)$. We define the oddly extended functions to be $\phi_{odd}(x)$ and $\psi_{odd}(x)$. We seek solution $u(x, t)$ to wave equation for $x \in \mathbb{R}$ so that it satisfies

$$u(x, 0) = \phi_{odd}(x) \quad , \quad u_t(x, 0) = \psi_{odd}(x) \qquad \text{for} \quad x \in \mathbb{R} \tag{29}$$

From d'Alembert formula, the solution is

$$u(x, t) = \frac{1}{2} \left[\phi_{odd}(x + ct) + \phi_{odd}(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{odd}(y) dy \tag{30}$$

We verify that $u(0, t) = 0$ is indeed satisfied by this expression. Also, using energy arguments, we can prove uniqueness of soluion satisfying (26)-(28). Hence, desired $v(x, t)$ is given by (30).

Now the formula (30) can be re-expressed in terms of $\phi(x)$ and $\psi(x)$, but the expression is different in different regimes of (x, t).

First, for $x > c|t|$, we notice that each of the arguments $x - ct$ and $x + ct$ are positive. Hence, (30) reduces to

$$v(x, t) = \frac{1}{2} \left[\phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy \tag{31}$$

The second regime is $0 < x < ct$. We have for $\phi_{odd}(x - ct) = -\phi(ct - x)$, and $\psi_{odd}(y) = -\psi(-y)$ for $y < 0$. Hence

$$\begin{aligned} v(x, t) = \frac{1}{2} \left[\phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \left[\int_{0}^{x+ct} \psi(y) dy + \int_{x-ct}^{0} [-\psi(-y)] \right] dy \\ = \frac{1}{2} \left[\phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \left[\int_{ct-x}^{x+ct} \psi(y) dy \right] \end{aligned} \tag{32}$$

Graphically, the result (32) can be interpreted as follows. Draw the backward characteristics from the point (x, t) (see Fig.2). In the regime $0 < x < ct$, such a backward characteristic intersects the t-axis, before crossing the x-axis at $(x - ct, 0)$. The formula (32) shows that the reflection induces a change of sign. The value of $v(x, t)$ now depends on the values of ϕ at the pair of points $ct \pm x$ and on ψ over the interval $(ct - x, ct + x)$, which is shorter than $(x - ct, x + ct)$. This is because the integral of $\psi_{odd}(y)$ over the symmetric interval $(x - ct, ct - x)$ is zero.

If $0 < x < -ct$, using similar arguments, we can check that (30) reduces to

$$v(x, t) = \frac{1}{2} \left[-\phi(-ct - x) + \phi(-ct + x) \right] - \frac{1}{2c} \left[\int_{-ct-x}^{-ct+x} \psi(y) dy \right] \tag{33}$$

The case of *Neumann problem* on a half-line for the wave equation is very similar, with an even extension of each of ϕ and ψ.

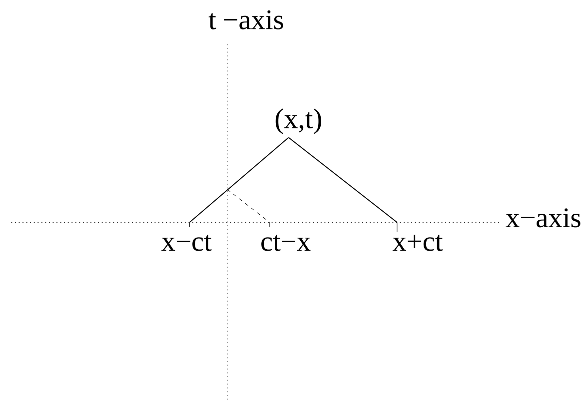

Figure 2: Backward characteristics from (x, t) for $0 < x < ct$

4 Wave Equation in a finite line

Consider the 1-D wave equation on a finite interval in x with homogeneous *Dirichlet* boundary conditions, that would correspond to say a guitar string with fixed ends. The initial value problem is given by:

$$v_{tt} = c^2 v_{xx}, \quad v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x) \text{ for } 0 < x < l \tag{34}$$

and

$$v(0,t) = v(l,t) = 0 \tag{35}$$

We can extend the data for each of $\phi(x)$ and $\psi(x)$ by first doing an *odd* extension to get a function over $(-l, l)$ and then define periodical extensios $\phi_{ext}(x)$ and $\psi_{ext}(x)$ over $\mathbb{R}$. Thus

$$\phi_{ext}(x) = \phi(x) \text{ for } x \in (0,l) \quad , \quad \phi_{ext}(x) = -\phi(-x) \text{ for } x \in (-l,0) \text{ and } \phi_{ext}(x+2l) = \phi_{ext}(x) \tag{36}$$

and a similar formula is valid for $\psi_{ext}(x)$. It is then possible to write solution as

$$v(x,t) = \frac{1}{2}\left[\phi_{ext}(x+ct) + \phi_{ext}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)ds \tag{37}$$

It can be checked directly that this satisfies $v(0,t) = 0$. To check $v(l,t) = 0$, we note that $\phi_{ext}(l-ct) = \phi_{ext}(-l-ct) = -\phi_{ext}(l+ct)$. Again, using periodicity and oddness of ψ_{ext}, we can prove that $\int_{l-ct}^{l+ct} \psi(y)dy = 0$.

While the expression (37) for the solution is relatively simple in terms of ϕ_{ext} and ψ_{ext}, it is much more complicated if we want to write it in terms of ϕ and ψ. There are different regimes of expression, depending on how many times reflection was possible at the end points $x = 0$ and $x = l$. I will refer you to Fig. 4 of the text on page 62 for a graphical illustration.

We will find later in this course alternate expressions of solution in terms of a Fourier Series.

5 Diffusion with a source

In this section, we solve the *inhomogeneous* diffusion equation in $\mathbb{R}^n$, for any dimension $n \geq 1$. So, the initial value problem of interest is

$$u_t - \kappa \Delta u = f(\mathbf{x}, t) \tag{38}$$

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \tag{39}$$

where ϕ, $f \in \mathbf{C}^0$. We will prove that the solution of the problem (38)-(39) is given by

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t)\phi(\mathbf{y}) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \tag{40}$$

This is an example of application of so-called *Duhammel's* principle where solution to inhomogeneous linear autonomous differential equation is expressed in terms of appropriate solution to the *homogeneous* solution, which in this case is $\mathcal{S}$. See text page 65 for analogy to ODEs.

It is convenient to define

$$v(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \tag{41}$$

Then noting

$$\lim_{\tau \to t^-} \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} = f(\mathbf{x}, t)$$

we obtain

$$v_t - \kappa \Delta v = \lim_{\tau \to t^-} \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} [\mathcal{S}_t(\mathbf{x} - \mathbf{y}, t - \tau) - \kappa \Delta_{\mathbf{x}} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau)] = f(\mathbf{x}, t) \tag{42}$$

Further, since

$$|\int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) f(\mathbf{y}, \tau) d\mathbf{y}| \leq \sup_{\mathbf{y} \in \mathbb{R}^{\mathbf{n}}} |f(\mathbf{y}, \tau)| \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) d\mathbf{y} = \|f(., \tau)\| \quad \text{for } \tau \ < \ t$$

it follows $\lim_{t \to 0^+} v(\mathbf{x}, t) = 0$. Since, we know from before that

$$w(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t)\phi(\mathbf{y} d\mathbf{y}$$

solves the heat equation without any source $w_t - \kappa \Delta w = 0$ and initial condition $w(\mathbf{x}, 0) = \phi(\mathbf{x})$, it follows that $u(\mathbf{x}, t) = v(\mathbf{x}, t) + w(\mathbf{x}, t)$ given by (40) solves the *inhomogeneous* heat equation (38) and satisfies the initial condition (39).

6 Inhomogenous Heat Equation in a half space $\mathbb{H}$

Note that the text on page 67 talks about solving inhomogenous diffusion equation on a half-line. There is no problem extending this idea to n-dimensions. We define half space

$$\mathbb{H} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_1 > 0\} \tag{43}$$

Our problem is to solve

$$v_t - \kappa \Delta v = f(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \mathbb{H} \tag{44}$$

$$v(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad \text{for } \ \mathbf{x} \in \mathbb{H} \tag{45}$$

$$v(\mathbf{x}, t) = 0 \quad \text{for } \ \mathbf{x} \in \partial\mathbb{H} \tag{46}$$

First, we note using odd-extension of $\phi(\mathbf{x})$ in the component x_1, as in 1-D, we can obtain a *source* type solution that satisfies the boundary condition on $x_1 = 0$ (which is the same as $\partial\mathbb{H}$. It is convenient to define

$$\mathbf{y}_- = (-y_1, y_2, y_3, ..., y_n) \quad , \quad \text{where } \mathbf{y} = (y_1, y_2, ..., y_n)$$

Then, it is easy to verify that

$$\mathcal{T}(\mathbf{x}, y, t) \equiv \mathcal{S}(\mathbf{x} - \mathbf{y}, t) - \mathcal{S}(\mathbf{x} - \mathbf{y}_-, t) \tag{47}$$

is a solution of the heat equation in the half-place $\mathbb{H}$ corresponding to a unit source at $\mathbf{y}$ that satisfies the homogenous *Dirichlet* condition (36). Therefore, using the same ideas as in the last section, we can verify that the solution to the problem posed in (44)-(46) is given by

$$v(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{T}(\mathbf{x}, \mathbf{y}, t)\phi(\mathbf{y})d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \mathcal{T}(\mathbf{x}, \mathbf{y}, t-\tau) f(\mathbf{y}, \tau) d\mathbf{y} d\tau \tag{48}$$

7 Note on Inhomogenous Dirichlet or Neumann conditions

So far, our boundary conditions, either for the heat or wave equation involved homogeneous *Dirichlet* or *Neumann* boundary conditions. However, solution for an inhomogenous condition is not a serious problem. For the sake of being definite, consider for instance

$$u_t - \kappa u_{xx} = f(x, t); \quad \text{for } \ x > 0 \quad , \quad u(0, t) = h(t) \quad , u(x, 0) = \phi(x) \tag{49}$$

We assume $h \in \mathbf{C}^1$. Noting that the function $w(x, t) \equiv e^{-x} h(t)$ satisfies the boundary condition at $x = 0$ and well-behaved in x as $x \to +\infty$, it follows that by decomposing $u(x, t) = w(x, t) + v(x, t)$, $v(x, t)$ satisfies

$$v_t - \kappa v_{xx} = f(x, t) - w_t + \kappa w_{xx} \equiv \tilde{f}(x, t) \tag{50}$$

$$v(0, t) = 0 \quad ; \quad v(x, 0) = \phi(x) - h(0) e^{-x} \equiv \tilde{\phi}(x) \tag{51}$$

Equations (38)-(39) defines an inhomogeneous heat equation with homogeneous *Dirichlet* condition and is of the type for which we have a general representation of solution, as seen in the last section.