Week 5 Lectures, Math 6451, Tanveer

1 Separation of variable method

The method of separation of variable is a suitable technique for determining solutions to linear PDEs, usually with constant coefficients, when the domain is bounded in at least one of the independent variables. We illustrate this procedure for 1-D wave equation and 2-D heat equation for Dirichlet, Neumann and Robin boundary conditions, though the idea is equally applicable for diffusion equation and Laplace's equation, and other constant coefficient equations.

The idea of separation of variable is first to seek simple solution to the PDE in the form of a product, each term in the product depending on only one independent variable. Solutions are then constrained by boundary conditions. This results in a countably infinite set of solutions. A linear superposition of such solution is also a solution, because of the linearity of the problem. As we shall find later, such linear superposition is capable of describing all reasonable initial conditions.

1.1 Dirichlet Problem

Consider

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad 0 < x < l \tag{1}$$

$$u(0,t) = 0 = u(l,t) \tag{2}$$

with initial condition

$$u(x,0) = \phi(x) \quad ; \quad u_t(x,0) = \psi(x) \tag{3}$$

Recall, there is a representation of this solution by doing odd extension about $x = 0$ to the interval $(-l, 0)$, and then periodically extending this problem (with period $2l$), and then using D'Alembert representation of solution, as you will see in last week notes. Uniqueness follows from application of energy method, as we saw earlier. Here, we are seeking a different form of the same solution.

We seek simple solution to (1), ignoring initial conditions (3) for now, in a product form $u(x,t) = X(x)T(t)$ Plugging it into (1) and dividing it the resulting equation by $c^2 X(x)T(t)$, we obtain

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} \tag{4}$$

If we set $t = 1$, for instance, then the left side of (4) is a number independent of x. Hence the right side of (4) cannot depend on x at all. Similarly, if we set x to some specific value, the right side of (4) is a constant; hence the left side (4) must be independent of t. Therefore, we conclude both left and right side of each side of (4) is some constant λ. Therefore,

$$T''(t) + \lambda c^2 \lambda T(t) = 0 \tag{5}$$

and

$$X''(x) + \lambda X(x) = 0 \tag{6}$$

The boundary conditions (2) imply that $X(0) = 0$ and $X(l) = 0$.

If $\lambda < 0$, then the solution to (6) that satisfies $X(0) = 0$ is given by

$$X(x) = C \sinh\left(\sqrt{-\lambda}x\right) \tag{7}$$

Since the function sinh does not vanish anywhere except the origin, any nontrivial solution in the form (7) is incapable of satisfying $X(l) = 0$. Therefore, we must discard the possibility of $\lambda < 0$.

If $\lambda = 0$, then the solution to (6) that satisfies $X(0) = 0$ is

$$X(x) = Cx, \tag{8}$$

But, this is not capable of satisfying $X(l) = 0$, unless $C = 0$, which corresponds to the trivial solution. Therefore, we conclude that $\lambda \neq 0$.

We are only left with the possibility $\lambda = \beta^2 > 0$. Then, solution to (6) satisfying $X(0) = 0$ is given by

$$X(x) = C \sin \beta x \tag{9}$$

In order for the solution to be nontrivial, *i.e.* $C \neq 0$, and yet $X(l) = 0$, we must have $\sin \beta l = 0$. Therefore $\beta l = n\pi$, for integer $n \neq 0$. We note that $\sin\left(\frac{-n\pi x}{l}\right) = -\sin\left(\frac{n\pi x}{l}\right)$. This implies that $n < 0$ does not generate an independent set of functions from what we obtain for $n > 0$. Thus, we can restrict to integer $n \geq 1$, and obtain

$$X(x) = C \sin\left(\frac{n\pi x}{l}\right) \equiv CX_n(x) \tag{10}$$

Corresponding to $\beta = \frac{n\pi}{l} \equiv \beta_n$, with $n > 1$, we may solve (5), with $\lambda = \beta_n^2$, to obtain

$$T(t) = A_n \cos \beta_n ct + B_n \sin \beta_n ct \equiv T_n(t) \tag{11}$$

where A_n and B_n are arbitary constants. Therefore, we obtain a countably infinite set of separable solutions to to (1) satisfying boundary conditions (2), indexed by integer n, and in the form $u(x,t) = X_n(x)T_n(t)$. A more general solution to (1), satisfying Dirichlet boundary condition (2) is of the form

$$u(x,t) = \sum_{n\geq 1} X_n(x)T_n(t) = \sum_{n\geq 1}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}, \tag{12}$$

assuming that this sum converges. We note that this solution (12) satisfies initial condition

$$u(x,0) = \sum_{n\geq 1} A_n \sin\frac{n\pi x}{l} \tag{13}$$

$$u_t(x,0) = \sum_{n\geq 1} \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l} \tag{14}$$

Therefore, if $\phi(x)$ and $\psi(x)$ in (3) is in the form of (13) and (14), the separation of variable method will have provided solution to IVP (1)-(3). We will later discover from the theory of Fourier Series, that this restriction on ϕ and ψ is rather mild–that *all* initial conditions of physical interest can be accomodated with solution (12) by suitably choosing A_n and B_n, depending on ϕ and ψ.

1.2 Neumann Condition

The same method works for Neumann problem as well. The text illustrates this for 1-D wave equation in section 4.2. We illustrate here for 2-D heat equation in a rectangular domain in $\mathbf{x} = (x, y)$. So, our problem of interest is

$$u_t - \kappa(u_{xx} + u_{yy}) = 0 \quad \text{for} \quad 0 < x < l, \; 0 < y < l \;, t > 0 \tag{15}$$

satisfying Neumann Boundary condition

$$u_x(0, y, t) = 0 = u_x(l, y, t) \quad , \quad u_y(x, 0, t) = 0 = u_y(x, l, t) \tag{16}$$

and initial condition

$$u(x, y, 0) = \phi(x, y) \tag{17}$$

Using energy method (see Homework 3 problem), we can prove that any solution satisfying (1)-(3) is unique. Here, we are showing existence.

As in the last subsection, first, we seek simple solution to (15) satisfying only (16) in the product form $u(x, y, t) = X(x)Y(y)T(t)$. Plugging into (15) and dividing the result by $X(x)Y(y)T(t)$, we obtain

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{T'(t)}{\kappa T(t)} \tag{18}$$

The left side depends only on x, while the right depends on (y, t). It follows that each side of the equation has to be some constant λ. Therefore,

$$-\frac{X''(x)}{X(x)} = \lambda \;, \quad \text{implying} \quad X''(x) + \lambda X(x) = 0 \tag{19}$$

The first set of boundary conditions in (16) imply that $X'(0) = 0$ and $X'(l) = 0$. However, if $\lambda < 0$, the solution satisfying $X'(0) = 0$ is given by $X(x) = C \cosh \sqrt{-\lambda} x$ which cannot satisfy $X'(l) = 0$ for nonzero C, since sinh is nonzero for nonzero argument. Therefore, we are left with only possibility of $\lambda = \beta^2 \geq 0$. In that case, a nontrivial solution to (19) satisfying $X'(0) = 0$ is

$$X(x) = \cos \beta x \tag{20}$$

In order to satisfy $X'(l) = 0$, we must have $\sin \beta l = 0$, therefore

$$\sqrt{\lambda} = \beta = \frac{n\pi}{l} \equiv \sqrt{\lambda_n} \tag{21}$$

and non-trivial solution for $X(x)$ is restricted to

$$X(x) = \cos \frac{n\pi x}{l} \equiv X_n(x) \quad \text{for } n \geq 0 \tag{22}$$

Note that $n < 0$ is excluded from the above set, since cos is an even function and replacing n by $-n$ does not generate any new set of solutions. However, unlike the Dirichlet problem, (21) allows the possibility of $n = 0$.

Having determined $X(x)$, we now determine $Y(y)$. For that purpose, it is convenient to rewrite (18) as

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{T'(t)}{\kappa T(t)} \tag{23}$$

and conclude that a function of y cannot be equal to a function of x and t unless each is some constant γ. Therefore,

$$-\frac{Y''(y)}{Y(y)} = \gamma \quad , \quad \text{implying} \quad Y''(y) + \gamma Y(y) = 0 \tag{24}$$

The second set of boundary conditions in (16) implies that

$$Y'(0) = 0 = Y'(l) \tag{25}$$

The determination of a nontrivial $Y(y)$ mirrors the procedure to determine $X(x)$. It is clear therefore, that $\gamma \geq 0$ and is restricted by the requirement

$$\sqrt{\gamma} = \frac{m\pi}{l} \equiv \sqrt{\gamma_m} \tag{26}$$

for integer $m \geq 0$, and Y is restricted to

$$Y(y) = \cos\frac{m\pi x}{l} \equiv Y_m(y) \tag{27}$$

Using (23), and relations (22) and (27), we obtain

$$T'(t) + \kappa\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)T(t) = 0 \tag{28}$$

Therefore, we must have $T(t)$ of the form

$$T(t) = A_{m,n}\exp\left[-\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)\kappa t\right] \equiv T_{m,n}(t) \tag{29}$$

for some constants $A_{m,n}$, which can depend on integers m, n. Therefore, using linear superposition of solutions of the form $X_n(x)Y_m(y)T_{mn}(t)$, we obtain a more general solution satisfying (15) and (16)

$$u(x,y,t) = \sum_{m\geq 0, n\geq 0} A_{m,n}\exp\left[-\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)\kappa t\right]\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} \tag{30}$$

This satisfies initial condition

$$u(x,y,0) = \sum_{m\geq 0, n\geq 0} A_{m,n}\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} \tag{31}$$

Therefore, if $\phi(x,y)$ in (17) is expressible in the form (31), we will have a representation of the solution to the Neumann problem, which is known to be unique from using energy method.. From Fourier series theory, which we will soon review, this is not much of a restriction on $\phi(x,y)$. Very general functions ϕ in the domain $(0,l) \times (0,l)$ have the representation (31) for suitably chosen $A_{m,n}$, which of course depends on ϕ.

2 Separation of variable in Circular Geometry

Separation of variable is not restricted to rectangular geometry. The method of solution provides solution to either Laplace, heat or wave equation and other related equations in circular (in 2-D) and spherical geometries (in 3-D). We illustrate this for Laplace's equation in a circle with Dirichlet boundary condition.

2.1 Laplace's equation in a circle

Consider the following 2-D problem:

$$\Delta u = 0 \quad ; \quad \text{for} \quad |\mathbf{x}| < 1 \tag{32}$$

with Dirichlet boundary condition on $|\mathbf{x}| = 1$. It is convenient in this case to introduce polar coordinates (r, θ). Using expression for Laplacian in polar coordinates, $u = u(r, \theta)$ satisfies

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{for} \quad r < 1 \tag{33}$$

The Dirichlet boundary condition is

$$u(1, \theta) = f(\theta) \tag{34}$$

We seek a separation of variable solution to (33) in the form

$$u(r, \theta) = R(r)\Theta(\theta) \tag{35}$$

Plugging into (33) and dividing the result by $R(r)\Theta(\theta)/r^2$, we obtain

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2 \left(\frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} \right) \tag{36}$$

Now, since left and right sides of (36) are functions of θ and r respectively, it follows that each side is some constant λ. Therefore, $\Theta(\theta)$, $R(r)$ satisfy:

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \tag{37}$$

$$R''(r) + \frac{1}{r} R'(r) = \frac{\lambda}{r^2} R(r) \tag{38}$$

Now, since the solution $u(r, \theta)$ sought is univalued in the disk of radius 1, it follows that $u(r, \theta + 2\pi) = u(r, \theta)$. Hence, we must have

$$\Theta(\theta + 2\pi) = \Theta(\theta) \tag{39}$$

Now, consider the case $\lambda < 0$. Then the solution to (37) is of the form

$$\Theta(\theta) = A \exp\left(-\sqrt{-\lambda}\theta\right) + B \exp\left(\sqrt{-\lambda}\theta\right) \tag{40}$$

But this does not satisfy (39) regardless of the value of A and B, except for the trivial case $(A, B) = (0, 0)$. Therefore, we rule this out.

For $\lambda = 0$, we have

$$\Theta(\theta) = A\theta + a_0 \tag{41}$$

This does not satisfy (39), unless $A = 0$. However, note that it does not rule out a nonzero a_0.

Now, comes the case of $\lambda = \mu^2 > 0$. Then, the solution to (37) is

$$\Theta(\theta) = a\cos\mu\theta + b\sin\mu\theta \tag{42}$$

Then condition (39) for any θ implies

$$\mu = m \quad > \ 0 \ \text{ an integer implying } \ \lambda = \lambda_m \equiv m^2 \tag{43}$$

Therefore,

$$\Theta(\theta) = \Theta_m(\theta) \equiv a_m \cos m\theta + b_m \sin m\theta \tag{44}$$

Since $m = 0$ in this formula includes the case of $\lambda = 0$, (44) provides *all* the eigen functions Θ_m for $\lambda = \lambda_m = m^2$ for integer $m \geq 0$. Having determined λ, we go back to (38) to obtain

$$R''(r) + \frac{1}{r}R'(r) - \frac{m^2}{r^2}R(r) = 0 \tag{45}$$

Noting that the equation is homogeneous in r, we look for solutions of the form $R(r) = r^\alpha$ and plug into (45), to discover that nontrival solutions are possible when $\alpha = \pm m$, *i.e* $R(r)$ is a linear combination of r^{-m} or r^m for $m > 0$. In the special case $m = 0$, we find $R(r) = c_0 + d_0 \log r$ for constants c_0 and d_0. However, since the solution $u(r,\theta)$ we are seeking is a classical solution without any singularities for $r < 1$, $R(r)$ must be well-behaved in particular at $r = 0$. This rules out r^{-m} term for $m > 0$ and $\log\ r$ for $m = 0$. Thus,

$$R(r) = R_m(r) \equiv r^m \ \text{ for integer } m \geq 0 \tag{46}$$

Taking a linear combination, we obtain for $r < 1$

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} \left(a_m r^m \cos m\theta + b_m r^m \sin\theta\right) \tag{47}$$

Assuming that this series converges, the solution is capable of satisfying Dirichlet boundary condition $u(1,\theta) = f(\theta)$ if

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos m\theta + b_m \sin m\theta\right) \tag{48}$$

As we shall see from the theory of Fourier Series, under mild restrictions, quite general functions $f(\theta)$ possess such an expansion, which is called its Fourier Series.

Note also, an alternate representation of the solution (65) in the form

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} c_m \cos(m\theta - \delta_m) \tag{49}$$

where $a_m = c_m \cos\delta_m$ and $b_m = c_m \sin\delta_m$. and this can be written as

$$u(r,\theta) = \Re\left\{\sum_{m=0}^{\infty} C_m z^m\right\}, \tag{50}$$

where $z = re^{i\theta}$, $\Re C_0 = a_0$ and $C_m = c_m e^{-i\delta_m}$. The quanitity on the right of (50) is the real part of an analytic function of $z = x + iy = re^{i\theta}$ and this representation is not surpising since any solution to Laplace's equation in 2-D, with (x, y) chosen as independent variables, is always either the real or imaginary part of an analytic function of $x + iy$.