Week 6 Lectures, Math 6451, Tanveer

1 Fourier Series

In the context of separation of variable to find solutions of PDEs, we encountered

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{for } x \in (0, l) \tag{1}$$

or

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{for } x \in (0, l) \tag{2}$$

and in other cases

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right\} \quad \text{for } x \in (-l, l) \tag{3}$$

The general representation (3) is called the Fourier representation of $f(x)$ in the $(-l, l)$ interval, while (1) and (2) are the Fourier sine and cosine representations of $f(x)$ in the interval $(0, l)$.

Our discussions revolve around some basic questions

1. When are represenations (1)-(3) valid and in what sense?
2. How do we determine coefficients a_n, b_n in terms of $f(x)$.
3. What conditions allow term by term differentiation of the Fourier-Series.

2 General $\mathcal{L}_2$ theory

We need enough generality to be able to lay the framework for discussion of more general series representations of $\mathcal{L}^2$, *i.e.* square integrable functions than (1)-(3), since they arise in other PDE problems.

In the space $\mathcal{L}_2(a, b)$ of generally complex valued functions in the interval (a, b), we introduce the $\mathcal{L}_2$ inner-product:

$$(f, g) = \int_a^b f(x) \bar{g}(x) dx \tag{4}$$

We note that the $\mathcal{L}^2$ norm is related through

$$\|f\| = (f, f)^{1/2} = \left[\int_a^b |f|^2(x) dx \right]^{1/2} \tag{5}$$

This may be generalized to any number of dimension, with x replaced by $\mathbf{x}$ and integration over the interval (a, b), replaced by integration over appropriate n-dimensional rectangle.

Definition 1 *A sequence* $\{X_n\}_{n=1}^{\infty} \in \mathcal{L}_2(a,b)$ *is orthogonal if*

$$(X_n, X_m) = 0 \quad \text{iff} \quad m \neq n \tag{6}$$

This sequence is said to be orthonormal, if in addition $(X_n, X_n) = \|X_n\|^2 = 1$.

Theorem 2 *Let* $\{X_n\}_{n=1}^{\infty} \in \mathcal{L}_2(a,b)$ *be a orthogonal set of functions. Let* $\|f\| < \infty$. *Let* N *be a fixed positive integer. The choice of* A_n *that minimizes mean square error* $E_N = \|f - \sum_{n=1}^{N} A_n X_n\|^2$ *is given by*

$$A_n = \frac{(f, X_n)}{\|X_n\|^2} \tag{7}$$

Further, we have

$$\|f\|^2 \geq \sum_{n=1}^{\infty} \frac{|(f, X_n)|^2}{\|X_n\|^2} \quad \text{Bessel inequality}$$

Proof. Define the square of the

$$E_N = \|f - \sum_{n=1}^{N} A_n X_n\|^2 = \left(f - \sum_{n=1}^{N} A_n X_n, f - \sum_{n=1}^{N} A_n X_n \right)$$

Expanding the above using properties of inner product and the orthogonality of X_n, we get

$$E_N = (f, f) - \sum_{n=1}^{N} A_n (X_n, f) - \sum_{n=1}^{n} A_n^* (f, X_n) + \sum_{n=1}^{N} A_n A_n^* (X_n, X_n) \tag{8}$$

We minimize E_n as a function of $2N$ real variables, $\{(c_n, d_n)\}_{n=1}^{N}$, where $A_n = c_n + id_n$. On taking partial derivatives, we get

$$\frac{\partial E_N}{\partial c_n} = -(X_n, f) - (f, X_n) + 2c_n (X_n, X_n) = 0 \quad \text{implying} \quad c_n = \frac{\Re\{(X_n, f)\}}{\|X_n\|^2}$$

Also,

$$\frac{\partial E_N}{\partial d_n} = -i(X_n, f) + i(f, X_n) + 2d_n (X_n, X_n) = 0 \quad \text{implying} \quad d_n = \frac{\Im\{(X_n, f)\}}{\|X_n\|^2}$$

Therefore $A_n = c_n + id_n$ is given by (7). Again with A_n given by (7), the corresponding E_N becomes

$$E_N = \|f\|^2 - \sum_{n \leq N} |A_n|^2 \|X_n\|^2 \geq 0$$

Taking the limit of $N \to \infty$, we obtain Bessel inequality. □

Theorem 3 *(Parseval's equality) The mean square error* E_N, *defined in Theorem 2,* $\to 0$ *as* $N \to \infty$, *if and only if*

$$\sum_{1}^{\infty} \frac{|(f, X_n)|^2}{\|X_n\|^2} = \|f\|^2 \quad ; \quad \text{Parseval equality} \tag{9}$$

When condition (9) holds for any function $f \in \mathcal{L}_2(a,b)$, *the orthogonal sequence* $\{X_n\}$ *is complete and forms a basis in* $\mathcal{L}_2(a,b)$.

Proof. From Theorem 2,

$$\lim_{N\to\infty} E_N = \|f\|^2 - \sum_1^\infty \frac{|(f, X_n)|^2}{\|X_n\|^2}$$

Therefore, it is zero iff and only if Parseval's equality holds. $\square$

Lemma 4 *The sequence*

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, ...\} \equiv \{X_n\}_{n=1}^\infty$$

is an orthogonal sequence in $\mathcal{L}_2(-\pi, \pi)$

Proof. This involves a simple calculation. First we note that for any n, $(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx dx = 0$ and also $(1, \cos nx) = 0$. Further, for $m \neq n$,

$$(\sin mx, \cos nx) = \int_{-\pi}^{\pi} \frac{1}{2} \left[\sin(m+n)x + \sin(m-n)x\right] dx = 0$$

while for $m \neq n$,

$$(cosmx, \cos nx) = \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(m-n)x + \cos(m+n)x\right] dx = 0$$

On the other hand for $m = n$,

$$(\sin mx, \cos nx) = (\sin nx, \cos nx) = \int_{-\pi}^{\pi} \frac{1}{2} \sin 2nx dx = 0$$

and

$$(\sin nx, \sin nx) = \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1}{2} \left(1 - \cos 2nx\right) dx = \pi$$

While

$$(\cos nx, \cos nx) = \int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{2\pi} \frac{1}{2} \left(1 + \cos 2nx\right) dx = \pi$$

$\square$

Corollary 5 *The sequence*

$$\left\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, ..\right\}$$

is an orthogonal sequence in $\mathcal{L}_2(-l, l)$.

Proof. This follows simply by introducing rescaled independent variable $\frac{x\pi}{l}$, which maps $(-l, l)$ to $(-\pi, \pi)$. We can then use the previous Lemma. $\square$

Remark 1 *By using a shift and a scaling, one can get a similar Fourier representations for square integrable functions in* (a, b).

Lemma 6 *For a $\mathcal{L}_2$ function f in $(-l,l)$ interval, the choice of a_m and b_m that minimizes the mean-square error*

$$\int_{-l}^{l}\left(f(x)-\frac{a_0}{2}-\sum_{m=1}^{M}a_m\cos\frac{m\pi x}{l}-\sum_{m=1}^{M}b_m\sin\frac{m\pi x}{l}\right)^2dx$$

for any positive integer M is given by

$$a_m=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{m\pi x}{l}dx\quad\text{for } 0\le m\le M$$

$$b_m=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{m\pi x}{l}dx\quad\text{for } 0\le m\le M$$

These are referred to as the Fourier-Coefficients of $f(x)$. Further,

$$l\left(|a_0|^2+\sum_{m=1}^{\infty}|a_m|^2+\sum_{m=1}^{\infty}|b_m|^2\right)\le\int_{-l}^{l}|f(x)|^2dx \tag{10}$$

PROOF. This follows simply by using the formula in Theorem 2 by identifying $(a,b)=(-l,l)$, $X_1=1$, $X_2=\cos x$, $X_3=\sin x$, $X_4=\cos 2x$, $X_5=\sin 2x$, etc and using values of integrals (X_n,X_n). The inequality (10) is simply a restatement of Bessel inequality for this case. □

Corollary 7 *For odd functions $f\in\mathcal{L}_2(-l,l)$, the cosine coefficients $a_m=0$, while for even square integrable functions, the sine coefficients $b_m=0$.*

PROOF. Proof follows simply by noting that if f is odd, then the formula that a_m involves integral of an odd function over $(-l,l)$, which is 0. When f is even, the same happens to the formula for b_m. □

Remark 2 *The above corollary implies that we do not need a theory for Fourier Sine and Fourier Cosine Series in (1) and (2), separate from the full Fourier Series (3), because we can do an odd or even extension of the function $f(x)$ to the interval $(-l,l)$. Then (3) will reduce to either (1) or (2) depending on whether the extension was odd or even.*

3 Pointwise convergence of Fourier Series

The primary purpose of this section is to prove the following Theorem:

Theorem 8 *(Pointwise Convergence of Fourier Series)*
Assume $f'(x^{\pm})$ exists at each point $x\in(-l,l)$ and that $f(x)$ is bounded in $[-l,l]$. Then,

1. *The Fourier Series (1) for f converges pointwise on $(-l,l)$ provided $f\in\mathbf{C}^0[-l,l]$.*
2. *More generally, is $f\in\mathcal{PC}^0[-l,l]$, i.e. continuous in the interval $[-l,l]$ except for a finite set of points, then the Fourier series converges at every point $x\in(-\infty,\infty)$. The sum is $\frac{1}{2}\left[f(x^+)+f(x^-)\right]$ for $x\in(-l,l)$ and $\frac{1}{2}\left[f_{ext}(x^+)+f_{ext}(x^-)\right]$ for $x\in(-\infty,\infty)$, where $f_{ext}(x)$ is the $2l$-periodic extension of $f(x)$.*

Remark 3 *The hypothesis for f' can be weakened even further, though it will not be necessary for the applications we have in mind.*

The proof of Theorem 8 will have to await some preliminary Lemmas. We will only prove it for the special case $l = \pi$, since the general case would follow merely by rescaling variable x, as seen before.

Definition 9 *For $f \in \mathcal{L}_2[-\pi, \pi]$ we define*

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N} (a_m \cos mx + b_m \sin mx) \tag{11}$$

where a_m and b_m are determined from

$$a_m = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{m\pi x}{l} dx \quad \text{for } 0 \leq m \leq M \tag{12}$$

$$b_m = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{m\pi x}{l} dx \quad \text{for } 0 \leq m \leq M \tag{13}$$

Lemma 10

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-y) f(y) dy \quad \text{where } K_N(\theta) = \frac{\sin\left[\left(N + \frac{1}{2}\right)\theta\right]}{\sin \frac{\theta}{2}} \tag{14}$$

Further,

$$\frac{1}{2\pi} \int_{0}^{\pi} K_N(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{0} K_N(\theta) d\theta = \frac{1}{2} \tag{15}$$

Proof. Using the formula for Fourier Coefficients, we get

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{m=1}^{N} (\cos my \cos mx + \sin my \sin mx)\right] f(y) dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x-y) f(y) dy$$

Therefore,

$$K_N(\theta) = 1 + 2 \sum_{n=1}^{N} \cos n\theta = \sum_{n=-N}^{N} e^{in\theta} = \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} = \frac{\sin\left[\left(N + \frac{1}{2}\right)\theta\right]}{\sin \frac{\theta}{2}} \tag{16}$$

Equation (15) follows simply by using the cosine series expansion of $K_N(\theta)$ in (16) and integrating term by term. All terms are zero except the first one, which gives a half. $\square$

Lemma 11 *Assume the same conditions on f as stated in Theorem 8. For $x \in (-l, l)$, define*

$$g_{\pm}(\theta) = \frac{f(x+\theta) - f(x^{\pm})}{\sin \frac{\theta}{2}} \tag{17}$$

Then, $\int_0^\pi |g_+(\theta)|^2 < \infty$ and $\int_{-\pi}^0 |g_-(\theta)|^2 < \infty$.

PROOF. A sufficient condition for each of these integrals to exist is for $\lim_{\theta\to 0^+} g_+(\theta)$ and $\lim_{\theta\to 0^-} g_-(\theta)$ to exist, since for other values of θ conditions on $f(x)$ make $g_\pm$ square integrable over the intervals $(0,\pi)$, $(-\pi,0)$ respectively. However, it is easy to see

$$\lim_{\theta\to 0^+} g_+(\theta) = \lim_{\theta\to 0^+} \frac{f(x+\theta)-f(x^+)}{\theta}\frac{\theta}{\sin\frac{\theta}{2}} = 2f'(x^+)$$

and

$$\lim_{\theta\to 0^-} g_-(\theta) = \lim_{\theta\to 0^-} \frac{f(x+\theta)-f(x^-)}{\theta}\frac{\theta}{\sin\frac{\theta}{2}} = 2f'(x^-)$$

each of which exists. $\square$

Proof of Theorem 8

We will assume $l=\pi$ since otherwise, we can rescale x. Note we only need to prove the second part, as the first part is a special case.

$$\begin{aligned} S_N(x) - \frac{1}{2}\left[f(x^+)+f(x^-)\right] &= \int_0^\pi \frac{K_N(\theta)}{2\pi}\left[f(x+\theta)-f(x^+)\right]d\theta + \int_{-\pi}^0 \frac{K_N(\theta)}{2\pi}\left[f(x+\theta)-f(x^-)\right]d\theta \\ &= \int_0^\pi g_+(\theta)\sin\left\{\left(N+\frac{1}{2}\right)\theta\right\}d\theta + \int_{-\pi}^0 g_+(\theta)\sin\left\{\left(N+\frac{1}{2}\right)\theta\right\}d\theta \qquad (18)\end{aligned}$$

It is easy to check that $\left\{\sin\left(N+\frac{1}{2}\right)\theta\right\}_{N=1}^{\infty}$ is an is an orthogonal set of functions either in the interval $(-\pi,0)$ or $(0,\pi)$. Therefore, from applying Bessel's inequality, it follows that each of the integrals on the right of (18) tend to zero $N\to\infty$, since Lemma 11 shows that g_+ and g_- are square integrable in $(0,\pi)$ and $(-\pi,0)$ respectively.

Further, for x outside the interval $(-\pi,\pi)$, it is clear that the Fourier Series is periodic and therefore converges to the periodically extended function $\frac{1}{2}\left[f_{ext}(x^+)+f_{ext}(x^-)\right]$.

Lemma 12 *If $f\in \mathbf{C}^0(-\infty,\infty)$ and is $2l$ periodic, and $f'(x^\pm)$ exists at every point $x\in(-\infty,\infty)$, then $S_N(x)$ converges to $f(x)$ pointwise.*

PROOF. We note that outside the $(-l,l)$, the extended function $f_{ext}(x)=f(x)$ itself. Since the function is continuous every where, $\frac{1}{2}\left[f_{ext}(x^+)+f_{ext}(x^-)\right]=f(x)$. From Theorem (8), the corollary follows. $\square$

Corollary 13 *If conditions of Theorem (8) hold, then the Fourier series (1) converges at the end points $\pm l$ to $f(\pm l)$ if and only if $f(-l)=f(l)$.*

PROOF. We simply note that when the condition $f(-l)=f(l)$, then the extended function $f_{ext}\in\mathcal{C}^0(-\infty,\infty)$ and its derivatives exists at every point in $(-\infty,\infty)$. Applying previous lemma, the concluson follows. $\square$

4 Uniform Convergence

We will now seek stronger conditions on f' so that $S_N(x)$ is uniformly convergent to f for $f\in\mathcal{C}^0(-l,l)$.

Lemma 14 *Assume f' exists at each point in $[-l,l]$ and that $f' \in \mathcal{L}_2(-l,l)$, and $f(l) = f(-l)$. Then, $S_N(x)$ converges to $f(x)$ uniformly in $[-l,l]$, i.e.*

$$\lim_{N\to\infty} \|f - S_N\|_\infty \to 0$$

PROOF. We will assume $l = \pi$ since otherwise we can introduce appropriate rescaled variable. The conditions on f' guarantees that

$$a'_m \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos mx\ dx$$

and

$$b'_m \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin mx\ dx$$

exists, and from Bessel's inequality

$$\pi\left[(a'_0)^2 + \sum_{m=1}^{\infty}\left\{(a'_m)^2 + (b'_m)^2\right\}\right] \le \int_{-\pi}^{\pi} |f'(x)|^2 < \infty$$

which implies that

$$\lim_{N\to\infty}\sum_{m=N+1}^{\infty}\left\{(a'_m)^2 + (b'_m)^2\right\} \to 0$$

On integration by parts, the Fourier coefficients of f itself becomes:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx = -\frac{1}{\pi m}\int_{-\pi}^{\pi} f'(x)\sin mx dx = -\frac{b'_m}{m}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx = \frac{1}{\pi m}\int_{-\pi}^{\pi} f'(x)\cos mx dx = \frac{a'_m}{m}$$

$$|f(x) - S_N(x)| \le \sum_{N+1}^{\infty} |a_m\cos mx + b_m \sin mx| \le \sum_{N+1}^{\infty}(|a_m| + |b_m|) \le \sum_{N+1}^{\infty}\frac{1}{m}\left(|a'_m| + |b'_m|\right)$$
$$\le \left(\sum_{N+1}^{\infty}\frac{1}{m^2}\right)^{1/2}\left\{\sum_{N+1}^{\infty}(a'_m)^2 + (b'_m)^2\right\}^{1/2} \to 0 \ \text{ as } \ N\to\infty \qquad (19)$$

Therefore $\lim_{N\to\infty}\|f - S_N\|_\infty = 0$. □

Remark 4 *If $f \in \mathbf{C}^0(-\infty,\infty)$ and $2l$ periodic (even without any condition on derivatives) it can be uniformly approximated in the maximum norm sense by a trigonometric polynomial, which is defined to be*

$$T(x) = \frac{c_0}{2} + \sum_{m=1}^{N}\left(c_n\cos\frac{n\pi x}{l} + d_n\sin\frac{n\pi x}{l}\right)$$

In order to show this, we first prove that such an f can be approximated well by a function $2l$-periodic $g \in \mathcal{C}^1(-\infty,\infty)$, which will allow us to use Lemma 14.

Lemma 15 *If $f \in \mathbf{C}^0(-\infty,\infty)$ and $2l$ periodic. For any $\epsilon > 0$, there exists a function $g \in \mathbf{C}^1(-\infty,\infty)$ and $2l$ periodic such that $\|f-g\|_\infty < \epsilon$.*

Proof. For each $\delta > 0$. Let

$$F_\delta(x) \equiv \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(y)dy = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t)dt$$

It is easily verified that F_δ is $2l$-periodic and $F_\delta \in \mathbf{C}^1(-\infty,\infty)$. Since $f(x) \in \mathbf{C}^0(-\infty,\infty)$ and $2l$ periodic, it is uniformly continuous. Given $\epsilon > 0$, there exists $\delta > 0$ independent of x, so that $|f(x+t) - f(x)| < \epsilon$ for $|t| < \delta$. Hence, if we choose such a δ and define $g(x) = F_\delta(x)$,

$$|g(x) - f(x)| \le \frac{1}{2\delta}|\int_{-\delta}^{\delta}(f(x+t) - f(x))dt| \le \frac{1}{2\delta}\int_{-\delta}^{\delta}|f(x+t) - f(x)|dt \le \epsilon$$

This is true for all x, hence Lemma follows. □

Theorem 16 *(Weirstrass Approximation Theorem) If $f \in \mathbf{C}^0(-\infty,\infty)$ and $2l$ periodic. For any $\epsilon > 0$, there exists a trigonometric polynomial $T(x)$ so that $\|f-T\|_\infty < \epsilon$.*

Proof. According to previous Lemma, for given ϵ, there exists $2l$-periodic $g \in \mathbf{C}^1(-\infty,\infty)$ so that $\|f-g\|_\infty < \frac{\epsilon}{2}$. Further, for such a g, Lemma 14 implies there exists $T(x)$, namely its Fourier series truncated to N terms for a sufficiently large N, so that $\|g-T\|_\infty < \frac{\epsilon}{2}$. Hence

$$\|f-T\|_\infty \le \|f-g\|_\infty + \|g-T\|_\infty < \epsilon$$

□

Theorem 17 *(Completeness of Fourier Series) The Fourier Series of any function $f \in \mathcal{L}_2[-l,l]$ converges to f in the $\mathcal{L}_2$ norm, i.e. in the mean-square sense and the Parseval's equality holds.*

Proof. It is known that $\mathcal{L}_2$ space is the completion of the set of $\mathbf{C}^0[-l,l]$ with the $\mathcal{L}_2$ norm. This means that that any $\epsilon > 0$, there exists $f_c \in \mathbf{C}^0[-l,l]$ so that $\|f - f_c\|_2 < \epsilon$, where $\|.\|_2$ denotes the $\mathcal{L}_2$ norm. We can also arrange $f_c(-l) = f_c(l)$. Now we apply periodic extension to f_c, so that $f_c \in \mathbf{C}^0(-\infty,\infty)$. Apply the Weirstrass approximation theorem, there exists trigonometric polynomial T so that $\|T - f_c\|_\infty < \frac{\epsilon}{\sqrt{2l}}$. Therefore, over the finite interval $[-l,l]$,

$$\|T - f_c\|_2 = \left[\int_{-l}^{l} |T(x) - F_c(x)|^2 dx\right]^{1/2} \le \epsilon$$

Therefore,

$$\|T - f\|_2 \le \|T - f_c\|_2 + \|f - f_c\|_2 \le 2\epsilon$$

Now, we know that for any trignometric polynomial with N terms the best approximation in the $\|.\|_2$ sense is through Fourier Series trucated to N term. If we denote this Fourier Polynomial by T_f, we get the result that

$$\|T_f - f\|_2 \le \|T - f\|_2 \le 2\epsilon$$

This is true for any ϵ. Hence Fourier Series converges to f in the mean-square sense. □

5 Non-Uniform Convergence near a point of discontinuity

Remark 5 *For piecewise continuous functions, near points of discontinuity, $S_N(x)$ does not approach $f(x)$ uniformly, even though it does so pointwise, except at the discontinuity itself. This is called the Gibbs phenomena.*

Theorem 18 *(Gibb's pheneomena) Assume $f \in \mathcal{PC}^0[-l,l]$ and $f'(x^{\pm})$ exists at each $x \in [-l,l]$. If there is a point of point of discontinuity $x_d \in (-l,l)$, then*

$$\lim_{N\to\infty} \sup_{x\in(x_d,l)} |S_N(x) - f(x)| \neq 0$$

$$\lim_{N\to\infty} \sup_{x\in(-l,x_d)} |S_N(x) - f(x)| \neq 0$$

Proof. We will only show the first part, since the proof of the second part proof is very similar. Also, for simplicity of notation, we take $l = \pi$, $x_d = 0$. We first consider the piecewise continuous function $h(x)$ defined as

$$h(x) = \frac{1}{2} \quad \text{for} \quad \pi > x > 0 \quad \text{and} \quad h(x) = -\frac{1}{2} \quad \text{for} \quad -\pi < x < 0$$

We can calculate its Fourier Coefficient and find its Fourier Series to be

$$\sum_{n=1,3,5..} \frac{2}{n\pi} \sin nx$$

We calculate

$$S_N(x) = \left(\int_0^{\pi} - \int_{-\pi}^{0}\right) K_N(x-y)\frac{dy}{4\pi} = \left(\int_0^{\pi} - \int_{-\pi}^{0}\right) \frac{\sin\left[(N+1/2)(x-y)\right]}{\sin\left[(x-y)/2\right]}\frac{dy}{4\pi}$$

Define $M = N + \frac{1}{2}$. In the first integral we take $\theta = M(x-y)$, and in the second integral, take $\theta = M(y-x)$. Then

$$S_N(x) = \left(\int_{M(x-\pi)}^{Mx} - \int_{-M(x+\pi)}^{-Mx}\right) \frac{\sin\theta}{2M\sin\left[\theta/(2M)\right]}\frac{d\theta}{2\pi}$$

$$\left(\int_{-Mx}^{Mx} - \int_{-M\pi+Mx}^{-M\pi-Mx}\right) \frac{\sin\theta}{2M\sin\left[\theta/(2M)\right]}\frac{d\theta}{2\pi} = \left(\int_{-Mx}^{Mx} - \int_{M\pi-Mx}^{M\pi+Mx}\right) \frac{\sin\theta}{2M\sin\left[\theta/(2M)\right]}\frac{d\theta}{2\pi}$$

Using calculus, we find that the first integral is maximized at $x = \frac{\pi}{M}$.

$$S_N\left(\frac{\pi}{M}\right) = \left(\int_{-\pi}^{\pi} - \int_{M\pi-\pi}^{M\pi+\pi}\right) \frac{\sin\theta}{2M\sin\left(\theta/(2M)\right)}\frac{d\theta}{2\pi}$$

The second integral tends to 0 as $M \to \infty$, since the denominator in the integrand is bounded below by M. Therefore, as $M \to \infty$,

$$S_N\left(\frac{\pi}{M}\right) \to \int_{-\pi}^{\pi} \frac{\sin\theta}{\theta}\frac{d\theta}{2\pi} = 0.59 \neq \frac{1}{2} \tag{20}$$

There is an overshoot.

We leave the details for a more general piecewise continuous function as an exercise, except to note that if $f(x)$ has a jump at 0, then if we decompose

$$f(x) = [f(0+) - f(0^-)]h(x) + \big(f(x) - [f(0+) - f(0^-)]h(x)\big)$$

then the term within (.) is continuous at $x = 0$, allowing use of previous theorems. $\square$

Remark 6 *In the following theorem, we consider a useful condition that allows a Fourier series to be differentiated term by term.*

Theorem 19 *(Differentiability of Fourier Series)*

If the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

has the property that

$$\sum_m m(|a_m| + |b_m|) < \infty$$

Then the Fourier Series can be differentiated term by term and the differentiated series converges uniformly to f'.

Proof. Leave it as an exercise. $\square$