Week 7 Lectures, Math 6451, Tanveer

In the next few week's lecture, we will consider Green's function for different linear differential equations. As we shall see, this is a very valuable constructive technique for determining solutions to certain boundary value problems. In the discussions, we will assume domain Ω to be an open bounded set, with a piecewise smooth boundary $\partial\Omega$. With appropriate assumptions on the decay of solutions, it is possible to take the limiting case of such domains as it becomes infinite size.

1 Green's function for Laplace's equation with Dirichlet condition

Definition 1 *The free space Green's function $G_0(|\mathbf{x}-\mathbf{x}_0|)$ is defined to be*

$$G_0(r) = \frac{1}{2\pi} \ \ln r \quad \text{for } n = 2$$

$$G_0(r) = -\frac{1}{4\pi r} \quad \text{for } n = 3$$

$$G_0(r) = -\frac{r^{2-n}\Gamma\left(\frac{n}{2}\right)}{(n-2)2\pi^{n/2}} \quad \text{for} \quad n > 3$$

Lemma 2 $G_0(|\mathbf{x}-\mathbf{x}_0)|)$ *satisfies* $\Delta G_0 = 0$ *for* $\mathbf{x} \neq \mathbf{x}_0$*, for any fixed* $\mathbf{x}_0$*. Further, for any* R*,*

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} \frac{\partial G_0}{\partial n} d\mathbf{x} = 1$$

while for continuous function $f(\mathbf{x})$ *continuous at* $\mathbf{x}_0$*,*

$$\lim_{\delta\to 0^+} \int_{|\mathbf{x}-\mathbf{x}_0|=\delta} f(\mathbf{x}) \frac{\partial G_0}{\partial n} d\mathbf{x} = f(\mathbf{x}_0)$$

Proof. We use radial coordinate r centered about $\mathbf{x}_0$. So $r = |\mathbf{x}-\mathbf{x}_0|$. Using expression for Laplacian is radial coordinates we get

$$\Delta G_0 = \frac{d^2}{dr^2} G_0(r) + \frac{n-1}{r} \frac{d}{dr} G_0(r) = 0$$

for $r \neq 0$, as can be readily verified from expression for $G_0(r)$. Further, we note that

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} \frac{\partial G_0}{\partial n} d\mathbf{x} = \left\{ \frac{dG_0}{dr} \right\}_{r=R} R^{n-1} \ \times \ (\text{Area of } \ \text{n}-\text{dimensional unit sphere}) \ = 1$$

Given $\epsilon > 0$, there exists δ so that if $r < \delta$, then $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. Then, using the above result on the integral $\frac{dG_0}{dr}$, we obtain $|\frac{\partial G_0}{\partial n}| = \frac{dG_0}{dr}$,

$$\int_{|\mathbf{x}-\mathbf{x}_0|=\delta} |f(\mathbf{x} - f(\mathbf{x}_0)||\frac{\partial G_0}{\partial n}| \leq \epsilon$$

Therefore,

$$|f(\mathbf{x}_0) - \int_{|\mathbf{x}-\mathbf{x}_0|=\delta} \frac{\partial G_0}{\partial n} f(\mathbf{x}) d\mathbf{x}| \leq \int_{|\mathbf{x}-\mathbf{x}_0|=\delta} \frac{\partial G_0}{\partial n} |f(\mathbf{x}) - f(\mathbf{x}_0)| d\mathbf{x} \leq \epsilon,$$

and the third statement of the Lemma follows. $\square$

Remark 1 *In the context of electro-statics, $G_0(|\mathbf{x}-\mathbf{x}_0|)$ for $n=3$ corresponds to the Coulomb potential due to a point charge in free-space located at a point $\mathbf{x}_0$. For $n=2$, $G_0(|\mathbf{x}-\mathbf{x}_0|)$ corresponds to the potential created by a line charge at $\mathbf{x}_0$ in free-space.*

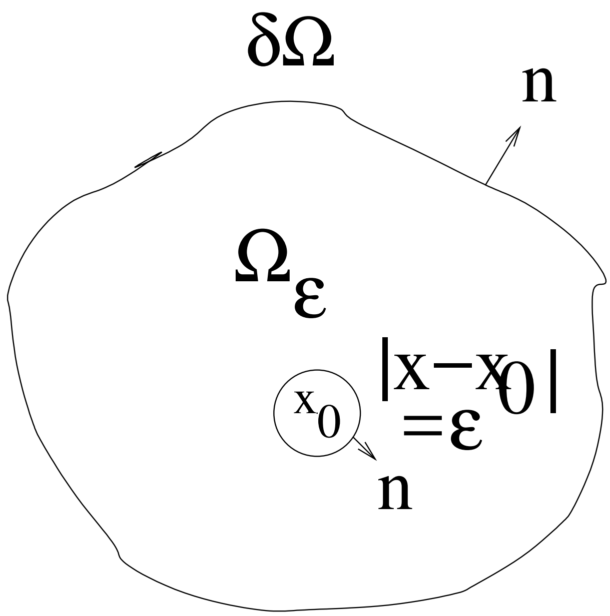

Figure 1: Domain Ω_ϵ with ϵ ball around $\mathbf{x}_0$ excluded

Lemma 3 *(Green's identity) For any $u \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^1(\bar{\Omega})$, we have for any $\mathbf{x}_0 \in \Omega$,*

$$u(\mathbf{x}_0) = -\int_{\partial\Omega} \left\{ G_0(|\mathbf{x}-\mathbf{x}_0|) \frac{\partial u}{\partial n}(\mathbf{x}) - u(\mathbf{x}) \frac{\partial G_0(|\mathbf{x}-\mathbf{x}_0|)}{\partial n} \right\} d\mathbf{x} + \int_{\Omega} G_0(|\mathbf{x}-\mathbf{x}_0|) \Delta u(\mathbf{x}) d\mathbf{x}$$

Proof. Define Ω_ϵ to the domain Ω with a sphere of radius ϵ around $\mathbf{x}_0$ excluded (See Fig. 1).

Then, since G_0 is a solution to Laplace's equation in the domain Ω_ϵ, we obtain

$$\int_{\Omega_\epsilon} \Delta u(\mathbf{x}) G_0(|\mathbf{x}-\mathbf{x}_0|)d\mathbf{x} = \int_{\Omega_\epsilon} \{\Delta u(\mathbf{x})\ G_0(|\mathbf{x}-\mathbf{x}_0|) - u(\mathbf{x})\ \Delta G_0(|\mathbf{x}-\mathbf{x}_0|)\}\, d\mathbf{x}$$
$$= \int_{\Omega_\epsilon} \nabla\cdot(\nabla u\ G_0(|\mathbf{x}-\mathbf{x}_0|) - u\ \nabla G_0(|\mathbf{x}-\mathbf{x}_0|))\, d\mathbf{x}$$
$$= -\int_{\partial\Omega} \left\{ u\frac{\partial G_0(|\mathbf{x}-\mathbf{x}_0|}{\partial n} - G_0(|\mathbf{x}-\mathbf{x}_0|)\frac{\partial u}{\partial n}\right\} d\mathbf{x} - \int_{|\mathbf{x}-\mathbf{x}_0|=\epsilon} \left\{ \frac{\partial u}{\partial n} G_0(|\mathbf{x}-\mathbf{x}_0|) - u\frac{\partial G_0(|\mathbf{x}-\mathbf{x}_0|)}{\partial n}\right\} d\mathbf{x} \tag{1}$$

Now, consider first the last term on the right

$$-\int_{|\mathbf{x}-\mathbf{x}_0|=\epsilon} \left\{ \frac{\partial u(\mathbf{x})}{\partial n} G_0(|\mathbf{x}-\mathbf{x}_0|) - u(\mathbf{x})\frac{\partial G_0(|\mathbf{x}-\mathbf{x}_0)|}{\partial n}\right\}$$

As $\epsilon \to 0^+$ since $u(\mathbf{x})$ is continuous at $\mathbf{x}_0$, we know from last Lemma that the second term is simply $u(\mathbf{x}_0)$. On the otherhand, the first term above tends to zero as $\epsilon \to 0^+$ since $r^{n-1}G_0(r) \to 0$ as $r \to 0$, while $\frac{\partial u}{\partial n}$ is continuous at $\mathbf{x}_0$. We also notice that

$$\lim_{\epsilon\to 0^+} \int_{|\mathbf{x}-\mathbf{x}_0|<\epsilon} G_0(|\mathbf{x}-\mathbf{x}_0)|)\nabla u d\mathbf{x} = 0$$

since $\lim_{r\to 0} r^{n-1}G_0(r) = 0$ and local volume element $d\mathbf{x}$ near $\mathbf{x} = \mathbf{x}_0$ scales as r^{n-1}. Therefore, combing all these results in (1), the Lemma follows. $\square$

Corollary 4 *If $\phi \in \mathbf{C}^\infty(\Omega)$ with compact support $\mathcal{K}$ contained in Ω, then*

$$\phi(\mathbf{x}_0) = \int_\Omega G_0(|\mathbf{x}-\mathbf{x}_0|)\Delta\phi(\mathbf{x} d\mathbf{x}$$

Proof. This follows immediately from last theorem, once we realize that for such functions ϕ, both ϕ and its derivatives are all 0 on $\partial\Omega$. $\square$

Definition 5 *The Green's function $G(\mathbf{x}, \mathbf{x}_0)$ of the operator $-\Delta$ in a domain $\Omega \subset \mathbb{R}^n$ is defined as a function that satisfies the following conditions:*

1. *$G(\mathbf{x}, \mathbf{x}_0)$ satisfies $\Delta G(\mathbf{x}, \mathbf{x}_0) = 0$ for any $\mathbf{x} \neq \mathbf{x}_0$ in Ω.*
2. *$G(\mathbf{x}, \mathbf{x}_0) = 0$ for $\mathbf{x} \in \partial\Omega$.*
3. *The function $G(\mathbf{x}, \mathbf{x}_0) - G_0(|\mathbf{x}-\mathbf{x}_0|)$ as a function of $\mathbf{x}$ is in $\mathbf{C}^2(\Omega) \cap \mathbf{C}^1(\bar{\mathbf{\Omega}})$ including at $\mathbf{x} = \mathbf{x}_0$.*

Remark 2 *In the context of electro-statics, for $n = 3$, $G(\mathbf{x}, \mathbf{x}_0)$ is the electro-static potential created by a point charge $\mathbf{x}_0$ in a domain Ω where the boundary $\partial\Omega$ is maintained at zero potential (by earthing for instance).*

Lemma 6 *(Symmetry of Green's function) For any domain Ω, the Green's function is symmetric, i.e.*

$$G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$$

Proof. Consider the domain Ω_ϵ shown in Figure 2 that has two small holes, each of radius ϵ and centered at $\mathbf{a}$ and $\mathbf{b}$ cut out from Ω. Let $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$ and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$. It is clear that each of u, v are harmonic in Ω_ϵ and hence

$$0 = \int_{\Omega_\epsilon} \{u\Delta v - v\Delta u\}\, d\mathbf{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\mathbf{x} + A_\epsilon + B_\epsilon \tag{2}$$

where

$$A_\epsilon = -\int_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\mathbf{x}$$

$$B_\epsilon = -\int_{|\mathbf{x}-\mathbf{b}|=\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\mathbf{x}$$

We know

$$\lim_{\epsilon\to 0)} A_\epsilon = \lim_{\epsilon\to 0^+} \int_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left([G_0(|\mathbf{x}-\mathbf{a}|) + H(\mathbf{x})] \frac{\partial v}{\partial n} - v\frac{\partial [H(\mathbf{x}) + G_0(|\mathbf{x}-\mathbf{a}|]}{\partial n} \right) d\mathbf{x} = v(\mathbf{a})$$

from using Lemma 2 and the fact that contribution from the Harmonic part H is zero, in the limit of surface measure shrinks to zero. Similarly,

$$\lim_{\epsilon\to 0^+} B_\epsilon = -u(\mathbf{b}),$$

the minus sign results from the fact that (2) is anti-symmetric on interchange of u and v. Therefore, using (2), and using the fact that both u and v are zero on $\partial\Omega$, it follows that

$$0 = \lim_{\epsilon\to 0^+} \{A_\epsilon + B_\epsilon\} = v(\mathbf{a}) - u(\mathbf{b}) = G(\mathbf{a}, \mathbf{b}) - G(\mathbf{b}, \mathbf{a})$$

□

Remark 3 *In electrostatics, the above Theorem is referred to as the principle of reciprocity. It asserts that a charge located at point* $\mathbf{a}$ *produces at point* $\mathbf{b}$ *the same potential as at point* $\mathbf{a}$ *due to a point charge located at* $\mathbf{b}$.

Theorem 7 *The solution of the problem*

$$\Delta u = f \quad \text{for} \quad \mathbf{x} \in \Omega \quad \text{, with} \quad u = h \quad \text{on} \quad \partial\Omega$$

for bounded $f \in \mathbf{C}^(\Omega)$, $h \in \mathbf{C}^0(\partial\Omega)$ is given by

$$u(\mathbf{x}_0) = \int_{\partial\Omega} h(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_0) d\mathbf{x} + \int_\Omega f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}$$

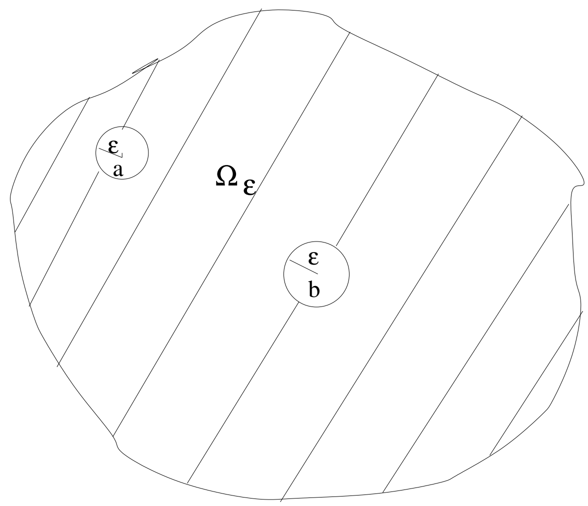

Figure 2: Domain Ω_ϵ in showing symmetry of Green's function

PROOF. We note that if we define $H(\mathbf{x},\mathbf{x}_0) = G(\mathbf{x},\mathbf{x}_0) - G_0(|\mathbf{x}-\mathbf{x}_0|)$, then H is harmonic in $\mathbf{x}$ in Ω and we therefore obtain

$$0 = -\int_{\partial\Omega} \left\{ H(\mathbf{x},\mathbf{x}_0)\frac{\partial u}{\partial n}(\mathbf{x}) - u(\mathbf{x})\frac{\partial H(\mathbf{x},\mathbf{x}_0)}{\partial n} \right\} d\mathbf{x} + \int_{\Omega} H(\mathbf{x},\mathbf{x}_0)\Delta u(\mathbf{x}) d\mathbf{x}$$

Adding this to the expression in Lemma 3, we obtain

$$u(\mathbf{x}_0) = \int_{\Omega} G(\mathbf{x},\mathbf{x}_0)\Delta u(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} \left\{ u(\mathbf{x}\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} - G(\mathbf{x},\mathbf{x}_0)\frac{\partial u(\mathbf{x})}{\partial n} \right\} d\mathbf{x} \tag{3}$$

Using the boundary condition on G, u on $\partial\Omega$ and the fact that $\Delta u = f$, we obtain the statement in the Lemma. □

Corollary 8 *A harmonic function u in Ω is completely determined in terms of its boundary data through an integral over $\partial\Omega$, as given in the last Theorem for $f = 0$.*

Remark 4 *We already know from the uniqueness theory of Laplace's equation with Dirichlet data that the expression given in the last theorem is the only solution to the given boundary value problem. The problem here is to find expression for Green's function $G(\mathbf{x},\mathbf{x}_0)$. This cannot be done explicitly, except for some simple domains illustrated in the next subsection. In 2-D ($n = 2$), $G(\mathbf{x},\mathbf{x}_0)$ is the real part of the log of a conformal mapping function that maps the domain Ω into a unit disk with $\mathbf{x}_0$ corresponding to the origin.*

1.1 Half-Space and Sphere

We solve for the harmonic functions in a half-space or on a sphere by determining Green's function through the method of reflection.

Half-Space

Consider the Green's function in a half-space domain

$$\mathbb{H}^+ = \{(x_1, x_2, x_3, ..x_n) : x_n > 0\}$$

The construction of Green's function can be done with the method of images as shown below. We can think of $G(\mathbf{x}, \mathbf{y})$ as the Coulomb potential due to a point positive charge at at $\mathbf{x} = \mathbf{y} = (y_1, y_2, ..y_n)$ when the plane boundary $x_n = 0$ is maintained at zero potential. We imagine a negative point charge outside the domain $\mathbb{H}^+$ located at $\tilde{\mathbf{y}} = (y_1, y_2, ...y_{n-1}, -y_n)$. Note that the actual positive charge and its image are at the same distance from the boundary $x_n = 0$. Hence,

$$G(\mathbf{x}, \mathbf{y}) = G_0(|\mathbf{x} - \mathbf{y}|) - G_0(|\mathbf{x} - \tilde{\mathbf{y}}|)$$

We can check that at $x_n = 0$,

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 \ + \ .. \ + \ (0 - y_n)^2} = |\mathbf{x} - \tilde{\mathbf{y}}|$$

and hence $G = 0$ as required. Further, for $\mathbf{x} \neq \mathbf{y}$,

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \Delta_{\mathbf{x}} G_0(|\mathbf{x} - \mathbf{y}|) - \Delta_{\mathbf{x}} G_0(|\mathbf{x} - \tilde{\mathbf{y}}|) = 0$$

and $G(\mathbf{x}, \mathbf{y}) - G_0(|\mathbf{x} - \mathbf{y}|) = -G_0(|\mathbf{x} - \tilde{\mathbf{y}}|)$ is obviously regular for any point $\mathbf{x} \in \mathbb{H}^+$.

In 3-D, the explicit expression for half-space Green's function is given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x} - \hat{\mathbf{x}}_\mathbf{0}|}$$

In 2-D, it is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log \frac{|\mathbf{x} - \mathbf{x}_0|}{|\mathbf{x} - \hat{\mathbf{x}}_\mathbf{0}|}$$

Green's function for Sphere in any dimension:

Given a fixed point $\mathbf{x}_0$, we define the reflected point $\mathbf{x}_0^* = \frac{a^2 \mathbf{x}_0}{|\mathbf{x}_0|^2}$. This point is outside the sphere and in the same line connecting the origin to $\mathbf{x}_0$. For a point $|\mathbf{x}| = a$, we note that

$$|\mathbf{x} - \mathbf{x}_0^*|^2 = (\mathbf{x} - \mathbf{x}_0^*) \cdot (\mathbf{x} - \mathbf{x}_0^*) = a^2 - 2\mathbf{x} \cdot \mathbf{x}_0^* + \mathbf{x}_0^* \cdot \mathbf{x}_0^*$$

$$= a^2 - 2\frac{a^2 \mathbf{x} \cdot \mathbf{x}_0}{|\mathbf{x}_0|^2} + \frac{a^4}{|\mathbf{x}_0|^2} = \frac{a^2}{|\mathbf{x}_0|^2} |\mathbf{x} - \mathbf{x}_0|^2$$

Therefore, since the above calculation shows the ratio of distances to the sphere is $|\mathbf{x}| = a$ of points $\mathbf{x}_0^*$ and $\mathbf{x}_0$ is a constant $\frac{a}{|\mathbf{x}_0|}$, it follows that

$$G(\mathbf{x}, \mathbf{x}_0) = G_0(|\mathbf{x} - \mathbf{x}_0|) - G_0\left(\frac{|\mathbf{x}_0|}{a} |\mathbf{x} - \mathbf{x}_0^*|\right)$$

since the two terms exactly cancel out.

When $n = 2$, we obtain

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log\left(\frac{|\mathbf{x} - \mathbf{x}_0|}{|\mathbf{x} - \mathbf{x}_0^*|}\right)$$

For $n = 3$,

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{a}{4\pi|\mathbf{x}_0||\mathbf{x} - \mathbf{x}_0^*|}$$

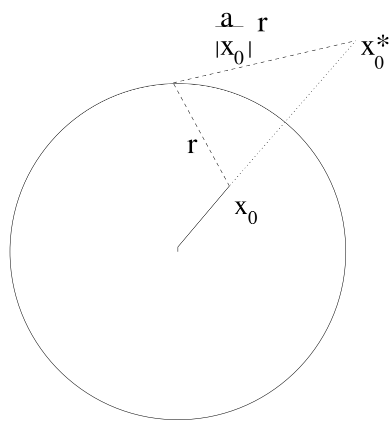

Figure 3: Ratio of distances from sphere of charge at $\mathbf{x}_0$ and image at $\mathbf{x}_0^*$ is a constant $\frac{a}{|\mathbf{x}_0|}$

Lemma 9 *The solution to $\Delta u = 0$ for $|\mathbf{x}| < a$ in 3-D with $u = \phi$ on $|\mathbf{x}| = a$ is given*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{4\pi a} \int_{|\mathbf{y}|=\mathbf{a}} \frac{h(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^3} d\mathbf{y}$$

If we express $\mathbf{x} = (r, \theta, \phi)$ and $\mathbf{y} = (a, \theta', \phi')$ in spherical coordinates, then the above implies,

$$u(r, \theta, \phi) = \frac{a(a^2 - r^2)}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\phi(\theta', \phi')}{(a^2 + r^2 - 2ar\cos\Psi)^{3/2}} \sin\theta' d\theta' d\phi'$$

where Ψ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$. In 2-D, the solution is given by

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{y}|=a} \frac{h(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^2} d\mathbf{y}$$

In polar coordinates, this becomes

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\theta')}{a^2 - 2ar\cos(\theta - \theta') + r^2} d\theta'$$

PROOF. The lemma follows from expressions for Green's function $G(\mathbf{x}, \mathbf{y})$ for $n = 3$ and $n = 2$, once we use Theorem 7 with $f = 0$. The details are left as an exercise. □

2 Distribution

We have already introduced Green's function in the context of Laplace's equation. With a view to generalizing this to other PDEs, we introduce the concept of a distribution.

Definition 10 *A function $\phi \in \mathbf{C}^\infty(\Omega)$ with compact support, i.e. $\phi = 0$ outside a bounded subset of Ω will be called a test function. The set of all such test functions will be denoted by $\mathcal{D}(\Omega)$. Note that Ω can be unbounded in this definition.*

Example: Consider

$$\phi_{\mathbf{y},\epsilon}(\mathbf{x}) = \exp\left(-\frac{\epsilon^2}{\epsilon^2 - |\mathbf{x}-\mathbf{y}|^2}\right) \quad \text{for} \quad |\mathbf{x}-\mathbf{y}| < \epsilon \quad \text{and} \quad \phi_{\mathbf{y},\epsilon}(\mathbf{x}) = 0 \quad \text{otherwise}$$

Definition 11 *Let $\{\phi_n\}_{n=1}^{\infty}$ be elements of $\mathcal{D}(\Omega)$. We say ϕ_n converges to ϕ in $\mathcal{D}(\Omega)$, if there is a compact set $\mathcal{K} \subset \Omega$ such that the supports of all ϕ_n lie in $\mathcal{K}$ and moreover, ϕ_n and all the derivatives of ϕ_n or arbitrary order converge uniformly to those of ϕ*

Lemma 12 *Let $\mathcal{K}$ be a compact set of Ω and let $f \in \mathbf{C}(\Omega)$ and have support contained in $\mathcal{K}$. For $\epsilon > 0$, let*

$$f_\epsilon(\mathbf{x}) = \frac{1}{C(\epsilon)} \int_{\mathcal{K}} \phi_{\mathbf{y},\epsilon}(\mathbf{x}) f(\mathbf{y}) d\mathbf{y}$$

where

$$C(\epsilon) = \int_{\mathbb{R}^m} \phi_{\mathbf{y},\epsilon}(\mathbf{x}) d\mathbf{x}$$

If $\epsilon < dist(\mathcal{K}, \partial\Omega)$, then $f_\epsilon \in \mathcal{D}(\Omega)$; more over $f_\epsilon \to f$ uniformly in $\mathbf{x}$ as $\epsilon \to 0$.

Proof. Exercise. □

Definition 13 *A distribution f is a linear mapping from $\mathcal{D}$ to $\mathbb{R}$, whose action on a test function ϕ denoted by (f, ϕ), has the property that if $\phi_n \to \phi$ in $\mathcal{D}$, then*

$$\lim_{n\to\infty} (f, \phi_n) = (f, \phi)$$

The space of distribution in a domain $\Omega \subset \mathbb{R}^n$ will be denoted by $\mathcal{D}'(\Omega)$.

Remark 5 *With the definition of distribution, we can uniquely associate a distribution f_g to every integrable function g, by defining $(f_g, \phi) = \int_{\mathbf{\Omega}} g(\mathbf{x})\phi(\mathbf{x})$. In such cases, for convenience of notation, we will blur the distinction between g and f_g and just use the notation g.*

Example 1: Suppose we consider a functional that assigns to a test function ϕ its value $\phi(\mathbf{0})$. This is clearly a bounded linear functional. This functional is denoted by δ and is called Dirac distribution or Dirac delta"function", though it is actually a functional. In the notation above,

$$(\delta, \phi) = \phi(\mathbf{0})$$

A generalization of this is $\delta_{\mathbf{x}}$, where

$$(\delta_{\mathbf{x}}, \phi) = \phi(\mathbf{x})$$

Example 2: In 1-D, the functional that assigns to ϕ the value $\phi''(5)$ is also a distribution. It is a linear functional since $(a\phi + b\psi)''(5) = a\phi''(5) + b\psi''(5)$; further it is continuous and therefore bounded since $\phi_n \to \phi$ in the sense of test function immediately implies $\phi_n''(5) \to \phi''(5)$. This functional will be denoted by δ_5'', as it will turn out to be the second derivative of δ_5 in the sense of distribution.

Lemma 14 *Let $f \in \mathcal{D}'(\Omega)$ and let K be a compact subset of $\Omega \subset \mathbb{R}^n$. There exists some $n \in \mathbb{N}$ and constant C so that for every $\phi \in \mathcal{D}$ with support in K,*

$$\Big|(f, \phi)\| \leq C \sum_{|\boldsymbol{\alpha}| \leq n} \max_{\mathbf{x} \in K} \Big| D^{\boldsymbol{\alpha}} \phi \Big| ,$$

where multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\Big|\boldsymbol{\alpha}\Big| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^{\boldsymbol{\alpha}} \phi = \partial^{|\boldsymbol{\alpha}|}_{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}} \phi$$

Proof. Assume on the contrary, there are are no such bounds. This means that for any choice $C = m \in \mathbb{N}$, there exists corresponding integer n and test function η_n so that

$$\Big|(f, \eta_n)\Big| > m \sum_{|\boldsymbol{\alpha}| \leq n} \max_{\mathbf{x} \in K} \Big| D^{\boldsymbol{\alpha}} \eta_n$$

Clearly, we can choose n to be an increasing function of m, and we can thereby define its inverse $m = C_n$. We have the property that $\lim_{n \to \infty} C_n = \infty$. Now, for $\phi_n = \eta_n / \Big|(f, \eta_n)\Big|$, we note $\phi_n \to 0$ as $n \to \infty$. On the otherhand, from construction, $(f, \phi_n) = 1$. But from continuity of linear functional f, we must have $\lim_{n \to \infty} (f, \phi_n) = 0$, which is a contradiction. $\square$

Definition 15 *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of distribution, it is said to converge weakly to f, if*

$$(f_n, \phi) \to (f, \phi) \quad \text{as} \quad n \to \infty$$

Example The source function for diffusion equation:

$$\mathcal{S}(\mathbf{x}, t) = \left(\frac{1}{\sqrt{4 \kappa \pi t}} \right)^n \exp \left[-\frac{|\mathbf{x}|^2}{4 \kappa t} \right] \quad \text{for } t \ > \ 0$$

converges to $\delta_{\mathbf{0}}$ as $t \to 0^+$, since recall we showed earlier in the context of discussion of diffusion problem that

$$\lim_{t \to 0^+} \int_{\Omega} \mathcal{S}(\mathbf{x}, t) \phi(\mathbf{x}) d\mathbf{x} = \phi(0)$$

So

$$\mathcal{S}(\mathbf{x}, t) \to \delta(\mathbf{x}) \text{ weakly as } t \to 0^+$$

Example: Recall in the context of Fourier Series discussion the function

$$K_N(\theta) = 1 + 2 \sum_{n=1}^{N} \cos n\theta = \frac{\sin[(N + \frac{1}{2})\theta]}{\sin \frac{\theta}{2}}$$

We had proved before that

$$\int_{-\pi}^{\pi} K_N(\theta) \phi(\theta) d\theta = 2\pi \phi(0)$$

for any $\mathbf{C}^1$ function ϕ in $(-\pi, \pi)$. Therefore, as $N \to \infty$,

$$K_N \to 2\pi\delta \quad \text{weakly} \quad \text{in } (-\pi, \pi)$$

Definition 16 *In 1-D, for any distribution f, we define its derivative f' by the formula*

$$(f', \phi) = -(f, \phi') \quad \text{for all} \quad \phi \in \mathcal{D}(\mathbb{R})$$

We can extend this definition to any number of variables. For instance f_{x_j} is defined as:

$$(f_{x_j}, \phi) = -(f, \phi_{x_j}) \quad \text{for all} \quad \phi \in \mathcal{D}(\mathbb{R}^n)$$

Example: In 1-D,

$$(\delta', \phi) = -(\delta, \phi') = -\phi'(0)$$
$$(\delta_5'', \phi) = -(\delta_5', \phi') = (\delta_5, \phi'') = \phi''(5)$$

Example: Consider the Heaviside function $H(x)$:

$$H(x) = 1, \quad \text{for } x \ > 0, \quad \text{and} \quad H(x) = 0 \quad \text{for} \quad x < 0$$

Then, clearly for any test function

$$(H'(x), \phi) = -(H(x), \phi'(x)) = -\int_0^\infty \phi'(x)dx = \phi(0)$$

So, $H' = \delta$.
Example Consider the complex series $\sum_{n=-\infty}^{\infty} e^{inx}$. Recall, in the discussion of Fourier Series, we had

$$K_N(x) = \sum_{n=-N}^{N} e^{inx}$$

and we showed earlier that $K_N \to 2\pi\delta$ in $(-\pi, \pi)$ as $N \to \infty$. Thus, we get in the *weak sense*

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi\delta(x) \text{ for } x \in (-\pi, \pi)$$

Example Recall Corollary for compactly supported test function ϕ in Ω:

$$\phi(\mathbf{x}_0) = \int_\Omega G_0(|\mathbf{x} - \mathbf{x}_0|)\Delta\phi(\mathbf{x})d\mathbf{x}$$

Therefore,

$$\phi(\mathbf{x}_0) = (G_0, \Delta\phi) = (\Delta G_0, \phi)$$

It follows that in the sense of distribution,

$$\Delta G_0(|\mathbf{x} - \mathbf{x}_0|) = \delta_{\mathbf{x}_0}(\mathbf{x})$$

Thus, Green's function has an interpretation in terms of distribution.