Week 8 Lectures, Math 6451, Tanveer

1 Green's function as a distribution

1.1 Laplace Operator

For the Poisson-Problem with homogeneous boundary condition:

$$\Delta u = f \quad \text{for} \quad \mathbf{x} \in \Omega \quad , \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{1}$$

we know that

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) d\mathbf{x} \tag{2}$$

On the otherhand, if u is a test function with support inside Ω, we have from using corollary 4 of week 7 notes that

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) \Delta u(\mathbf{x}) d\mathbf{x} = (u, \Delta G(., \mathbf{x}_0)) \tag{3}$$

Therefore, in the sense of distribution,

$$\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{4}$$

Therefore, we view solution (24) as a principle of linear superposition. In the physical context ($n = 3$), it means that the potential caused by charge density f in a domain Ω with boundary at zero potential is given by a linear superposition of point charge potentials satisfying the same boundary conditions, with a weighting proportional to the infinitesimal charge $f(\mathbf{x}) d\mathbf{x}$ present in a volume element $d\mathbf{x}$ at $\mathbf{x}$.

Further, note that $G(\mathbf{x}, \mathbf{x}_0) = G_0(|\mathbf{x} - \mathbf{x}_0|) + H(\mathbf{x}, \mathbf{x}_0)$, where H is harmonic in $\mathbf{x}$. It follows that

$$\Delta G_0(|\mathbf{x} - \mathbf{x}_0|) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{5}$$

More generally if we have a linear constant coefficient PDE in the form

$$Lu = f \text{ , for } x \in \Omega \subset \mathbb{R}^n \text{ , with } u = 0 \text{ on } \partial\Omega \tag{6}$$

then if we can find Greens function satisfying

$$LG = \delta(\mathbf{x} - \mathbf{x}_0) \quad , \text{ with } G = 0 \text{ on } \partial\Omega \tag{7}$$

then, we can show that

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) d\mathbf{x} \tag{8}$$

1.2 Heat Equation

Recall from last class that the source function:

$$\mathcal{S} = \left(\frac{1}{4\pi\kappa t}\right)^{n/2} \exp\left[-\frac{|\mathbf{x}|^2}{4\kappa t}\right] \tag{9}$$

satisfies

$$\mathcal{S}_t = \kappa\Delta\mathcal{S} \quad \text{for} \quad \mathbf{x}\in\mathbb{R}^n \ , t>0 \quad , \text{with} \quad \mathcal{S}(\mathbf{x},0^+) = \delta(\mathbf{x}) \tag{10}$$

It can be shown (exercise) that

$$R(\mathbf{x},t) = \mathcal{S}(\mathbf{x}-\mathbf{x}_0, t-t_0) \quad \text{for} \quad t>t_0 \quad \text{and} \quad R(\mathbf{x},t)=0 \quad \text{for } t<t_0 \tag{11}$$

satisfies

$$R_t - \kappa\Delta R = \delta(\mathbf{x}-\mathbf{x}_0)\delta(t-t_0) \tag{12}$$

2 Eigen Function Expansion for Green's Function

2.1 Heat Equation

Consider Source solution to heat equation bounded domain $\Omega\subset\mathbb{R}^n$ with homogeneous Dirichlet Boundary conditions:

$$S_t = \kappa\Delta S \quad \text{for} \quad \mathbf{x}\in\Omega \quad \text{and} \quad S=0 \quad \text{on} \quad \partial\Omega, \quad \text{with } S(\mathbf{x},0) = \delta(\mathbf{x}-\mathbf{x}_0) \tag{13}$$

In terms of the $S(\mathbf{x},\mathbf{x}_0,t)$ the solution to the initial value problem

$$u_t = \kappa\Delta u \quad \text{for} \quad \mathbf{x}\in\Omega \quad \text{and} \quad u=0 \quad \text{on} \quad \partial\Omega, \quad \text{with } u(\mathbf{x},0) = \phi(\mathbf{x}) \tag{14}$$

is given by

$$u(\mathbf{x},t) = \int_\Omega S(\mathbf{x},\mathbf{y},t)\phi(\mathbf{y})d\mathbf{y} \tag{15}$$

On the otherhand, if we denote the orthonormalized eigenfunctions $\{X_n\}_{n=1}^\infty$ and corresponding eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the operator $-\Delta$ with homogenous boundary conditions on $\partial\Omega$, we know that solution to heat equation has the form

$$u(\mathbf{x},t) = \sum_{n=1}^\infty \exp\left[-\lambda_n\kappa t\right] X_n(\mathbf{x}) \tag{16}$$

where

$$c_n = (\phi, X_n) = \int_\Omega \phi(\mathbf{x})X_n(\mathbf{x})d\mathbf{x} \tag{17}$$

Then,

$$u(\mathbf{x},t) = \sum_{n=1}^\infty \left(\int_\Omega \phi(\mathbf{y})X_n(\mathbf{y})d\mathbf{y}\right)\exp\left[-\lambda_n\kappa t\right] X_n(\mathbf{x}) = \left(\int_\Omega \phi(\mathbf{y})\left\{\sum_{n=1}^\infty X_n(\mathbf{y})X_n(\mathbf{x})e^{-\lambda_n\kappa t}\right\}d\mathbf{y}\right) \tag{18}$$

Therefore, it follows that under the assumption that the summation converges absolutely and uniformly,

$$S(\mathbf{x},\mathbf{y},t) = \sum_{n=1}^\infty X_n(\mathbf{y})X_n(\mathbf{x})e^{-\lambda_n\kappa t} \tag{19}$$

Note, that this implies that in the sense of distribution, we must have

$$\delta(\mathbf{x}-\mathbf{y}) = \sum_{n=1}^\infty X_n(\mathbf{y})X_n(\mathbf{x})$$

This is true for any complete ortho-normal basis.

3 Fourier Transform

Notation: For multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, with each $\alpha_j \in \mathbb{N}$, it is convenient to introduce operator

$$D^\alpha \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} .... \partial_{x_n}^{\alpha_n}$$

The order of this operator is denoted by $|\alpha| \equiv \alpha_1 + \alpha_2 + .. + \alpha_n$.

Definition 1 *$\mathcal{S}(\mathbb{R}^n)$ be the space of all functions ϕ on $\mathbb{R}^n$ which are of the class $\mathbf{C}^\infty$ and such that for any integer $j \geq 0$, $|\mathbf{x}|^j |D^\alpha \phi| < \infty$, for $|\alpha| = j$. This is referred to usually as the Schwartz class of functions.*

Definition 2 *A tempered distribution in $\mathbb{R}^n$ is a continuous linear functional on the class of $\phi \in \mathcal{S}(\mathbb{R}^n)$.*

Remark 1 *Note that every tempered-distribution is a distribution, but the converse is not true.*

The Fourier transform of a continuous absolutely integrable function f on $\mathbb{R}^n$ is defined by

$$\hat{f}(\mathbf{k}) = \mathcal{F}[f](\mathbf{k}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left[-i\mathbf{k}\cdot\mathbf{x}\right] f(\mathbf{x}) d\mathbf{x}$$

In particular, this defines Fourier-Transform for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 3 *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. Moreover, the mapping is continuous from $\mathcal{S}(\mathbb{R}^n)$ to itself.*

Proof. We leave the proof to the reader. $\square$

Theorem 4 *Let $g \in \mathcal{S}(\mathbb{R}^n)$. Then there is a unique $f \in \mathcal{S}(\mathbb{R}^n)$ such that $g = \mathcal{F}[f]$. Futhermore, the inverse Fourier transform of g is given by*

$$f(\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}} g(\mathbf{k}) d\mathbf{k} \tag{20}$$

Proof. Let $Q_M = [-M, M]^n$, and let f be given by the above formula. Then, we find

$$\begin{aligned} \tilde{f}(\mathbf{k}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x} &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{x}} \int_{\mathbb{R}^n} e^{i\boldsymbol{\eta}\cdot\mathbf{x}} g(\boldsymbol{\eta}) d\boldsymbol{\eta} d\mathbf{x} \\ &= \lim_{M\to\infty} \int_{\mathbb{R}^n} \int_{Q_M} e^{i(\boldsymbol{\eta}-\mathbf{k})\cdot\mathbf{x}} g(\eta) d\mathbf{x} d\eta = \pi^{-n} \lim_{M\to\infty} \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{\sin M(\eta_i - k_i)}{\eta_i - k_i} g(\eta) d\eta \end{aligned}$$

However, it is easily seen that as $M \to \infty$, $\frac{\sin M(\eta_i - k_i)}{\eta_i - k_i} \to \pi\delta(\eta_i - k_i)$ in the sense of distribution. Therefore, it follows that

$$\hat{f}(\mathbf{k}) = g(\mathbf{k})$$

An analogous caclulation shows that if $g = \hat{h}$ for some $h \in \mathcal{S}(\mathbb{R}^n)$, then $h = f$ as given by equation (20). $\square$

Theorem 5 *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $(\hat{f}, \hat{g}) = (f, g)$.*

PROOF. We have

$$(f,g) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \overline{g(\mathbf{x})}\int_{\mathbb{R}^n} \hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}d\mathbf{x}$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{f}(\mathbf{k})\int_{\mathbb{R}^n} \overline{g(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}}d\mathbf{x}d\mathbf{k} = \int_{\mathbb{R}^n} \overline{\hat{g}(\mathbf{k})}\hat{f}(\mathbf{k})d\mathbf{k} = (\hat{f}, \hat{g})$$

□

We now seek to give meaning to Fourier-Transform of tempered distribution.

Definition 6 *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then the Fourier transform of f is defined by the functional*

$$(\mathcal{F}[f], \phi) = (f, \mathcal{F}^{-1}[\phi]) \quad \text{for} \quad \phi \in \mathcal{S}(\mathbb{R}^n)$$

Remark 2 *It is not difficult to see that $\mathcal{F}$ is a continuous mapping from $\mathcal{S}'(\mathbb{R}^n)$ onto itself. The formulas for Fourier transform and its inverse still hold for tempered distribution.*

Example We want Fourier-transform of δ distribution. From definition

$$(\mathcal{F}[\delta], \phi) = (\delta, \mathcal{F}^{-1}[\phi]) = \mathcal{F}^{-1}[\phi](\mathbf{0}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \phi(\mathbf{x})d\mathbf{x}$$

Therefore $\mathcal{F}[\delta] = (2\pi)^{-n/2}$, a constant.
Example The above relation is symmetric. since the Fourier-transform of 1 equal to $(2\pi)^{n/2}\delta$ since

$$(\mathcal{F}[1], \phi) = (1, \mathcal{F}^{-1}[\phi]) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\phi](\mathbf{x})d\mathbf{x} = (2\pi)^{n/2}\mathcal{F}\mathcal{F}^{-1}[\phi](\mathbf{0}) = (2\pi)^{n/2}\phi(\mathbf{0})$$

Therefore, $\mathcal{F}[1] = (2\pi)^{n/2}\delta$
Example: Let $\delta(|\mathbf{x}| - a)$ represent a uniform mass distribution on a sphere of radius a, *i.e.*

$$(\delta(|\mathbf{x}| - a), \phi) = \int_{|\mathbf{x}|=a} \phi(\mathbf{x})dS$$

Then,

$$\mathcal{F}[\delta(|\mathbf{x}| - a)](\mathbf{k}) = (2\pi)^{-n/2}\int_{|\mathbf{x}|=a} e^{-i\mathbf{k}\cdot\mathbf{x}}dS$$

For $n = 3$, using spherical polar coordinates, we get

$$\mathcal{F}[\delta(|\mathbf{x}| - a)](\mathbf{k}) = (2\pi)^{-n/2}\int_0^{\pi}\int_0^{2\pi} e^{-ia\rho\cos\theta}\sin\theta d\phi d\theta = \sqrt{\frac{2}{\pi}}a\frac{\sin a|\mathbf{k}|}{|\mathbf{k}|}$$

4 Examples of Fourier-Transform of Distributions

Example: Using the same argument, as for $\mathcal{F}[1]$, except with $\mathbf{k}$ replaced by $\mathbf{k}-\boldsymbol{\eta}$, we find:

$$\mathcal{F}\left[\exp[i\boldsymbol{\eta}\cdot\mathbf{x}]\right](\mathbf{k}) = (2\pi)^{m/2}\delta(\mathbf{k}-\boldsymbol{\eta})$$

If f is a 2π-periodic distribution represented by a Fourier-series:

$$f(x) = \sum_{n=1}^{\infty} c_n e^{inx},$$

we find that

$$\mathcal{F}[f](k) = \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

Definition 7 *A distribution f is said to have a compact support, if there exists a compact $\mathcal{K}$ so that for all test function ϕ with support in $\mathbb{R}^n\backslash\mathcal{K}$, $(f,\phi)=0$. An example of this is $\delta(\mathbf{x})$, whose support is only $\{\mathbf{0}\}$.*

Example Let f be a distribution with compact support. Then for any $\phi \in \mathbf{C}^{\infty}(\mathbb{R}^n)$, we set $(f,\phi)=(f,\phi_0)$, where $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ and ϕ_0 agrees with ϕ is a neighborhood of the support of f. If ϕ_1 also has similar property as ϕ_0, it is clear from definition of f that $(f,\phi_0-\phi_1)=0$ since from construction, the support of $\phi_0-\phi_1$ is outside the support of f. Thus, (f,ϕ) can be defined unambigously (not depending on which ϕ_0 is used).

We claim that $\mathcal{F}[f]$ is the function

$$\mathcal{F}[f](\mathbf{k}) = (2\pi)^{-m/2}\left(\overline{f(\mathbf{x})}, e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

Here $(\bar{f},\phi)$ is defined as the complex conjugate of $(f,\bar{\phi})$. This follows since for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(f(\mathbf{x}), e^{i\mathbf{k}\cdot\mathbf{x}}\right)\phi(\mathbf{k})d\mathbf{k} = (f,\mathcal{F}^{-1}[\phi])) = (2\pi)^{-n/2}\left(f, \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k})d\mathbf{k}\right)$$

Example: Let $\delta(|\mathbf{x}|-a)$ represent a uniform mass distribution on a sphere of radius a, *i.e.*

$$(\delta(|\mathbf{x}|-a),\phi) = \int_{|\mathbf{x}|=a}\phi(\mathbf{x})dS$$

Then,

$$\mathcal{F}[\delta(|\mathbf{x}|-a)](\mathbf{k}) = (2\pi)^{-n/2}\int_{|\mathbf{x}|=a} e^{-i\mathbf{k}\cdot\mathbf{x}}dS$$

For $n=3$, using spherical polar coordinates, we get

$$\mathcal{F}[\delta(|\mathbf{x}|-a)](\mathbf{k}) = (2\pi)^{-n/2}\int_0^{\pi}\int_0^{2\pi} e^{-ia\rho\cos\theta}\sin\theta d\phi d\theta = \sqrt{\frac{2}{\pi}}a\frac{\sin a|\mathbf{k}|}{|\mathbf{k}|}$$

5 The Source Solution (fundamental solution) for the wave equation:

Consider solution to

$$S_{tt} = \Delta S \quad , \quad \text{for} \;\; \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}, \quad \text{with} \;\; S(\mathbf{x},0) = 0, \quad S_t(\mathbf{x},0) = \delta(\mathbf{x}) \tag{21}$$

Fourier-transforming, we obtain

$$\hat{\mathcal{S}}_{tt} = -\mathbf{k}^2 \hat{S} \quad \text{with} \;\; \hat{S}(\mathbf{k},0) = 0 \;\; \hat{S}_t(\mathbf{k},0) = (2\pi)^{-n/2} \tag{22}$$

Therefore,

$$\mathcal{S}(\mathbf{k},t) = (2\pi)^{-n/2} \frac{\sin|\mathbf{k}|t}{|\mathbf{k}|} \tag{23}$$

For $n = 3$, from one of the previous examples, it follows that

$$S(\mathbf{x},t) = \frac{\delta(|\mathbf{x}| - t)}{4\pi t} \tag{24}$$

This method of finding Green's function is generally valid for any constant coefficient system in free-space. For example, if we have have a PDE of the form

$$\mathcal{P}(\partial_{\mathbf{x}})G = \delta(\mathbf{x})$$

where $\mathcal{P}$ is a polynomial, then application of Fourier-Transform leads to an algebraic relation:

$$\hat{G}(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}\mathcal{P}(i\mathbf{k})}$$

We can then recover G by Fourier-transform. Note that since the above is true for any dimension, it can accomodate PDEs involving both t and $\mathbf{x}$, but just considering a higher dimensional variable $\overline{\mathbf{x}} = (\mathbf{x}, t)$.

6 Laplace Transform:

If $f \in \mathcal{S}'(\mathbb{R})$ have support contained in $\{x \geq 0\}$. Then obviously $e^{-\mu x} f(x)$ is also in $\mathcal{S}'(\mathbb{R})$ for every $\mu > 0$. Formally, we have

$$\mathcal{F}\left[e^{-\mu x} f\right](\mathbf{k}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) e^{-ikx} e^{-\mu x} dx = \mathcal{F}[f](k - i\mu)$$

Hence, it is sensible to define $\mathcal{F}[f](k - i\mu) = \mathcal{F}[f e^{-\mu x}]$. This defines $\mathcal{F}[f]$ in the lower half of the complex k-plane–as a generalized function of $\Re k$, depending of $\Im k$ as a paraemeter. Actually, however, this function is analytic k in the lower-half plane ($\Im k < 0$). The Laplace-transform is defined as

$$\mathcal{L}[f](s) = \sqrt{2\pi}\mathcal{F}[f](-is);$$

for $f \in \mathcal{S}'$ with support in $\{x \geq 0\}$, it is defined in the right half-plane complex plane ($\Re s \geq 0$). Formally, we have

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x) dx$$

If $f \notin \mathcal{S}'(R)$, but $e^{-\mu x} f \in \mathcal{S}'(\mathbb{R})$ for some $\mu > 0$, then we can define $\mathcal{L}[f]$ in the right-half plane $\Re\ s \geq \mu$. We note that by inverting the Fourier-transform we obtain

$$e^{-\mu x} f(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}[f](\mu + ik),$$

or equivalently,

$$f(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{sx} \mathcal{L}[f](s) ds$$

In using the above formula, we must ensure that the resulting expression vanishes for $x < 0$, since this was our basic assumption. Typically, one shows this by closing the contour of integration by a half-circle to the right; e^{sx} decays rapidly in the right half plane. For this argument to work, it is necessary to choose μ to the right of the singularities of f.

Example: Consider

$$u_t = u_{xx} \quad \text{ for } x \in (0,1), \quad t > 0 \;\text{ with }\; u(x,0) = 0, \quad u(0,t) = 1 = u(1,t) \;\text{ for } t > 0$$

Laplace transform in time leads to

$$s\mathcal{L}[u](x,s) = \mathcal{L}[u]_{xx}(x,s) \quad ; \quad \mathcal{L}[u](0,s) = \frac{1}{s} = \mathcal{L}[u](1,s)$$

The equation for the solution is

$$\mathcal{L}[u](x,s) = \frac{\cosh\left(\sqrt{s}\left(x - \frac{1}{2}\right)\right)}{s \cosh\left(\sqrt{s}/2\right)}$$

Using the inverse transform, we obtain

$$u(x,t) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{st} \frac{\cosh\left(\sqrt{s}(x - \frac{1}{2}\right)}{s \cosh\left(\sqrt{s}/2\right)} ds$$

The integral cannot be evaluated in closed form, however, through contour deformation, and change of variables $\sqrt{s} -> s_1$, it is possible to use calculus of residues (complex variable technique) and obtain a series form of solution.

7 Wave equation

7.1 Solution in higher dimension through Spherical Means

Assume u is a classical solution to the initial value problem for n-dimensional wave equation for $n \geq 2$:

$$u_{tt} - \Delta u = 0 \;\text{ for }\; \mathbf{x} \in \mathbb{R}^n \text{ for } t > 0 \;\text{ with }\; u(\mathbf{x}, 0) = \phi(\mathbf{x}) \;\;,\;\; u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \tag{25}$$

where $\phi \in \mathbf{C}^2$ and $\psi \in \mathbf{C}^1$. For $t > 0$, $r > 0$, we define $U(\mathbf{x}; r, t)$ to be the spherical average over the surface of an n-dimensional ball $B(\mathbf{x}, r)$ of radius r, centered at $\mathbf{x}$, and denoted by

$$U(\mathbf{x}; r, t) = \frac{1}{A_r} \int_{\partial B(\mathbf{x}; r)} u(\mathbf{y}, t) d\mathbf{y} \equiv \fint_{\partial B(\mathbf{x}; r)} u(\mathbf{y}, t) d\mathbf{y} \tag{26}$$

where A_r is the surface area of an n dimensional ball of radius r. Note $A_r = n\alpha(n) r^{n-1}$, where volume of the n-dimensional sphere is $\alpha(n) r^n$. It is to be noted that

$$\lim_{r \to 0^+} U(\mathbf{x}; r, t) = u(\mathbf{x}, t)$$

from continuity of $\mathbf{u}$. We can similarly define

$$G(\mathbf{x}; r) = \fint_{\partial B(\mathbf{x}; r)} \phi(\mathbf{y}, t) d\mathbf{y} \tag{27}$$

$$H(\mathbf{x}; r) = \fint_{\partial B(\mathbf{x}; r)} \psi(\mathbf{y}, t) d\mathbf{y} \tag{28}$$

For fixed $\mathbf{x}$, we regard U as a function of r and t. We claim

Lemma 8 *For fixed* $\mathbf{x}$*,* $U(\mathbf{x}; r, t)$ *is a solution of the initial value problem:*

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \;\text{ for }\; r > 0, \;\; t > 0 \;\text{ and }\; U(\mathbf{x}; r, 0) = G(\mathbf{x}, r), \;\; U_t(\mathbf{x}; r, 0) = H(\mathbf{x}, r) \tag{29}$$

Proof. For convenience, we depart from our usual convention and denote 'surface area' element on the n-dimensional ball as dS. Symbol $dS_{\mathbf{y}}$ will denote surface area element in the variable $\mathbf{y}$. We note that

$$U(\mathbf{x}; r, t) = \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t)) dS_{\mathbf{y}} = \fint_{\partial B(\mathbf{0}, 1)} u(\mathbf{x} + r\mathbf{z}, t)) dS_{\mathbf{z}} \tag{30}$$

Therefore,

$$\begin{aligned} U_r(\mathbf{x}; r, t) = \fint_{\partial B(\mathbf{0}, 1)} \mathbf{z} \cdot \nabla u(\mathbf{x} + r\mathbf{z}, t) dS_{\mathbf{z}} = \fint_{\partial B(\mathbf{x}, r)} \frac{\partial u}{\partial n} dS_{\mathbf{y}} = \frac{1}{A_r} \int_{\partial B(\mathbf{x}, r)} \frac{\partial u}{\partial n} dS_{\mathbf{y}} \\ = \frac{1}{A_r} \int_{B(\mathbf{x}, r)} \Delta u(\mathbf{y}, t) d\mathbf{y} == \frac{1}{n\alpha(n) r^{n-1}} \int_{B(\mathbf{x}, r)} \Delta u(\mathbf{y}, t) d\mathbf{y}, \end{aligned} \tag{31}$$

Thus, using (25), it follows that

$$U_r(\mathbf{x}; r, t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(\mathbf{x}, r)} u_{tt} d\mathbf{y} \tag{32}$$

and therefore,

$$\frac{\partial}{\partial r} \left\{ r^{n-1} U_r(\mathbf{x}; r, t) \right\} = \frac{1}{n\alpha(n)} \int_{\partial B(\mathbf{x}; r)} u_{tt}(\mathbf{y}, t) dS_{\mathbf{y}} = r^{n-1} U_{tt} \tag{33}$$

This gives the PDE for U given in the Lemma. Further, it is clear from definition of G and H that U satisfies the given initial conditions. $\square$

Theorem 9 *(Kirchoff Formula for $n = 3$)*

The solution to the initial value problem for the three-dimensional dimensional wave equation in free-space:

$$u_{tt} - \Delta u = 0 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3 \text{ for } t > 0 \quad \text{with} \quad u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad , \quad u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \tag{34}$$

is given by

$$u(\mathbf{x}, t) = \fint_{\partial B(\mathbf{x},t)} \{t\psi(\mathbf{y} + \phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla\phi(\mathbf{y})\} \, dS_{\mathbf{y}} \tag{35}$$

Proof.

If note that if we introduce transformation

$$\tilde{U}(\mathbf{x}; r, t) = rU(\mathbf{x}; r, t), \quad \tilde{G} = rG, \quad \tilde{H} = rH$$

Then, simple calculation shows

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad \text{for } r > 0 \quad , \ t > 0, \quad \text{with} \quad \tilde{U}(r, 0) = \tilde{G}(r), \quad \tilde{U}_t(r, 0) = \tilde{H}(r) \quad , \tilde{U}(0, t) = 0$$

This is the Wave equation on a half-line with a homogeneous *Dirichlet* condition. As discussed in Week 4 lectures (see equation (32) on page 4, with $c = 1$) for $0 \le r \le t$, we obtain

$$\tilde{U}(\mathbf{x}; r, t) = \frac{1}{2}\left[\tilde{G}(r + t) - \tilde{G}(t - r)\right] + \frac{1}{2}\int_{t-r}^{r+t} \tilde{H}(y) dy$$

Since $u(\mathbf{x}, t) = \lim_{r \to 0^+} \frac{\tilde{U}(\mathbf{x}; r, t)}{r}$,

$$u(\mathbf{x}, t) = \lim_{r \to 0^+} \left\{\frac{1}{2r}\left[\tilde{G}(r + t) - \tilde{G}(t - r)\right] + \frac{1}{2r}\int_{t-r}^{r+t} \tilde{H}(y) dy\right\}$$

$$= \tilde{G}'(t) + \tilde{H}(t) = \partial_t \left(t \fint_{\partial B(\mathbf{x},t)} \phi dS\right) + t \fint_{\partial B(\mathbf{x},t)} \psi dS = \partial_t \left(t \fint_{\partial B(\mathbf{0},1)} \phi(\mathbf{x} + t\mathbf{z}) dS_{\mathbf{z}}\right) + t \fint_{\partial B(\mathbf{0},1)} \psi dS_{\mathbf{z}}$$

$$= \fint_{\partial B(\mathbf{0},1)} \{t\psi(\mathbf{x} + t\mathbf{z}) + \phi(\mathbf{x} + t\mathbf{z}) + t\mathbf{z} \cdot \nabla\phi(\mathbf{x} + t\mathbf{z})\} \, dS_{\mathbf{z}} = \fint_{\partial B(\mathbf{x},t)} \{t\psi(\mathbf{y}) + \phi(\mathbf{y}) + \nabla\phi(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})\} \, dS_{\mathbf{y}}$$

□

Theorem 10 *(Poisson Formula for $n = 2$)*

The solution to the initial value problem for the two-dimensional dimensional wave equation in free-space:

$$u_{tt} - \Delta u = 0 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^2 \text{ for } t > 0 \quad \text{with} \quad u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad , \quad u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \tag{36}$$

is given by

$$u(\mathbf{x}, t) = \frac{1}{2} \fint_{B(\mathbf{x},t)} \frac{\left\{t\psi(\mathbf{y}) + t^2\psi(\mathbf{y} + t(\mathbf{y} - \mathbf{x}) \cdot \nabla\phi(\mathbf{y})\right\}}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{1/2}} d\mathbf{y} \tag{37}$$

Proof. We imbed the 2-D problem as part of 3-D problem. With $\bar{\mathbf{x}} = (x_1, x_2, x_3)$, $\mathbf{x} = (x_1, x_2)$, $\bar{u}(\bar{\mathbf{x}}, t) = u(\mathbf{x}, t)$. Then $\bar{u}$ satisfies the 3-D wave equation with initial condition

$$\bar{\phi}(\bar{\mathbf{x}}) = \phi(\mathbf{x}) \quad \text{and} \quad \bar{\psi}(\bar{\mathbf{x}}) = \psi(\mathbf{x})$$

Then, we have from the 3-D calculation,

$$\bar{u}(\bar{\mathbf{x}}, t) = \partial_t \left(t \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{\phi} d\bar{S} \right) + t \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{\psi} d\bar{S}$$

where $\bar{B}(\bar{\mathbf{x}}, t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{\mathbf{x}}$ of radius $t > 0$, and $d\bar{S}$ denotes the two dimensional surface measure. Now we observe that

$$\fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} d\bar{S} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} d\bar{S} = \frac{2}{4\pi t^2} \int_{B(\mathbf{x}, t)} g(\mathbf{y}) (1 + (\nabla \gamma)^2)^{1/2} d\mathbf{y}$$

where $\gamma(\mathbf{y}) = \sqrt{t^2 - |\mathbf{y} - \mathbf{x}|^2}$ for $\mathbf{y} \in B(\mathbf{x}, t)$. The factor 2 enters since $\partial \bar{B}(\bar{\mathbf{x}}, t)$ consists of two hemispheres. Computation shows that $[1 + (\nabla \gamma)^2]^{1/2} = t[t^2 - |\mathbf{y} - \mathbf{x}|^2]^{-1/2}$. Therefore,

$$\fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} d\bar{S} = \frac{t}{2} \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{\sqrt{t^2 - |\mathbf{y} - \mathbf{x}|^2}} d\mathbf{y} = \fint_{B(\mathbf{0}, 1)} \frac{g(\mathbf{x} + t\mathbf{z})}{\sqrt{1 - |\mathbf{z}|^2}} d\mathbf{z}$$

The rest of the theorem is straight-forward computation. □

7.2 Source solution for Wave Equation

We consider source solution $S(\mathbf{x}, t)$ that satisfies:

$$S_{tt} = c^2 \Delta S \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n \quad , \quad t \in \mathbb{R} \text{ with } \; S(\mathbf{x}, 0) = 0 \;\; ; S_t(\mathbf{x}, 0) = \delta(\mathbf{x}) \tag{38}$$

This is referred to as the *Riemann* problem. To find formula for S, let $\psi(\mathbf{x})$ be any test function and we define

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} S(\mathbf{x} - \mathbf{y}, t) \psi(\mathbf{y}) d\mathbf{y} \tag{39}$$

Then, assuming integration with respect to $\mathbf{x}$ and t commutes with the integration with respect to $\mathbf{y}$, it follows that u satisfies wave equation as well, and satisfies initial conditions

$$u(\mathbf{x}, 0) = 0, \quad \text{and} \quad u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \tag{40}$$

From D'Alembert formula, the solution to this for $n = 1$ is given by

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \psi(y) dy = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$$

Therefore, $S(x - y, t) = \frac{1}{2c}$ for $y - x \in (-ct, ct)$ and 0 otherwise. Therefore,

$$S(x, t) = \frac{1}{2c} \quad \text{for} \quad |x| < ct \quad \text{and} \quad 0 \quad \text{for} \quad |x| > ct \quad \text{for } t > 0 \tag{41}$$

Similar formula can be found for $t < 0$. Notice that if we replace t by $-t$ in the initial value problem, it only reverses the sign of ψ. Using Heaviside function H[1], we obtain

$$S(x,t) = \frac{1}{2c} H(c^2t^2 - x^2) sgn(t) \tag{42}$$

For 1-D, the *Riemann* function is actually a function in the usual sense. This is not the case in higher dimension, where it is a distribution.

Note from *Kirchoff*-formula that solution for $t > 0$ for $n = 3$ is given by

$$\frac{1}{4\pi c^2 t} \int_{\partial B(\mathbf{x},t)} \psi(\mathbf{y}) dS_{\mathbf{y}} = u(\mathbf{x},t) = \int_{\mathbb{R}^3} S(\mathbf{x}-\mathbf{y},t)\psi(\mathbf{y})d\mathbf{y} = \int_0^\infty dr \int_{\partial B(\mathbf{x},r)} S(\mathbf{y}-\mathbf{x},t)\psi(\mathbf{y})dS_{\mathbf{y}} \tag{43}$$

Therefore for $t > 0$, $S(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\delta(|\mathbf{x}| - ct)$. We can similarly analyze the solution for $t < 0$, noticing that that replacing t by $-t$ in the problem posed for S has the effect of switching its sign. Therefore, for $t < 0$, we must have $S(\mathbf{x},t) = -\frac{1}{4\pi c^2(-t)}\delta(|\mathbf{x}| + ct)$. A uniform expression is given by

$$S(\mathbf{x},t) = \frac{1}{2\pi c}\delta(|\mathbf{x}|^2 - c^2t^2) sgn(t)$$

In 2-D similar calculation using Poisson formula shows

$$S(\mathbf{x},t) = \frac{1}{2\pi c}\left(c^2t^2 - |\mathbf{x}|^2\right)^{-1/2} \text{ for } \; |\mathbf{x}| < ct \;\text{ and }\; = 0 \text{ otherwise}$$

[1]Recall $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$)