Week 9 Lectures, Math 6451, Tanveer

1 Maximum principle for Elliptic Equations

Recall, you proved the weak maximum principle for Laplace's equation as part of homework. We now wish to generalize this for elliptic equations. Our treatment here is close to Rogers and Renardy's PDE book, with some elaboration when needed.

$$Lu := \sum_{i,j} a_{i,j} \partial_{x_i x_j} u + \sum_i b_i \partial_{x_i} u + cu = 0 \tag{1}$$

where recall ellipticity implies

$$\sum_{i,j} a_{i,j} \xi_i \xi_j > 0 \tag{2}$$

for any vector $\boldsymbol{\xi} \neq 0$ in $\mathbb{R}^n$, which is equivalent to the matrix $(a_{i,j})$[1] having all positive eigenvalues. We will assume coefficients $a_{i,j}, b_i \in \mathbf{C}(\bar{\Omega})$ and $u \in \mathbf{C}^2(\Omega) \cap \mathbf{C}(\bar{\Omega})$. Note that the coefficients $a_{i,j}, b_i, c$ need not be constants but generally depends on $\mathbf{x}$.

Theorem 1 *Weak Maximum Principle Assume $Lu \geq 0$ ($Lu \leq 0$) in a bounded domain Ω where $c(\mathbf{x}) = 0$. Then its maximum (minimum) is achieved on $\partial\Omega$*

Proof. We will only prove the maximum principle for $Lu \geq 0$. The proof of the minimum principle for $Lu \leq 0$ will then follow simply by switching the sign of u. Assume $a_{1,1} > 0$.[2] We introduce associated function $u_\epsilon = u + \epsilon e^{\alpha x_1}$. If $Lu \geq 0$, then

$$Lu_\epsilon = Lu + \epsilon \left(\alpha^2 a_{11} + \alpha b_1 \right) > 0 \tag{3}$$

for choice of $\alpha > |b_1|/a_{11}$. On the otherhand, similar to arguments in Lemma 3, week 3 notes, ellipticity of L implies that at an interior maximum $\mathbf{x}_0$, we must have

$$Lu_\epsilon(\mathbf{x}_0) \leq 0 \ , \tag{4}$$

which is a contradiction. Hence

$$\sup_{\bar{\Omega}} u < \sup_{\bar{\Omega}} u_\epsilon < \sup_{\partial\Omega} u_\epsilon \leq \sup_{\partial\Omega} u + C\epsilon \ , \tag{5}$$

[1] Recall we arrange $a_{i,j}$ so that it is symmetric matrix and therefore has real eigenvalues

[2] Recall that normal coordinate transformations in section 4 of week 2 notes implies that this could always be arranged for elliptic equations with a transformation of variables

where $C = \sup_{\mathbf{x}\in\Omega} e^{\alpha x_1}$. Therefore,

$$\sup_{\bar{\Omega}} u < \sup_{\partial\Omega} u + C\epsilon \tag{6}$$

and taking the limit $\epsilon \to 0$, we arrive at the weak maximum principle. $\square$

Corollary 2 *Let Ω be bounded and assume $c \leq 0$ in Ω. Let $Lu \geq 0$ ($Lu \leq 0$). Then*

$$\sup_{\mathbf{x}\in\bar{\Omega}} u \leq \sup_{\mathbf{x}\in\partial\Omega} u^+ \quad \left(\inf_{\mathbf{x}\in\bar{\Omega}} u \geq \inf_{\mathbf{x}\in\partial\Omega} u^- \right) ,$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

Proof. Again we only prove the maximum principle for $Lu \geq 0$, since it is clear that the minimum principle for $Lu \leq 0$ follows from it by mere switching sign of u. If $u \leq 0$ in Ω, then there is nothing to prove since $u^+ \geq 0$ from definition. Otherwise, define $\Omega^+ = \Omega \cap \{u > 0\}$. Note that $u = 0$ on $\partial\Omega^+$. Furthermore, for $\mathbf{x} \in \Omega^+$, since $-cu \geq 0$,

$$a_{i,j}\partial_{x_i x_j} u + b_i \partial_{x_i} u \geq 0 \ , \tag{7}$$

and maximum principle holds for the elliptic operator $a_{i,j}\partial_{x_i x_j} + b_i\partial_{x_i}$ on Ω^+ implying

$$\sup_{\mathbf{x}\in\Omega^+} u \leq \sup_{\mathbf{x}\in\partial\Omega^+} u = 0 \tag{8}$$

Therefore, the maximum of u, if positive, is achieved on the boundary $\partial\Omega$. $\square$

Remark 1 The following statement follows from the from the weak maximum principle and the above corollary and is useful in proving uniqueness of solution to $Lu = f$ for given u on $\partial\Omega$.

Corollary 3 *Let Ω be bounded and $c \leq 0$. If $Lu = Lv$ in Ω and $u = v$ on $\partial\Omega$, then $u = v$ in Ω. If $Lu \leq Lv$ in Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

2 Strong Maximum Principle for Elliptic Equations

We will first prove the following preliminary Lemma that will be used to establish the strong maximum principle.

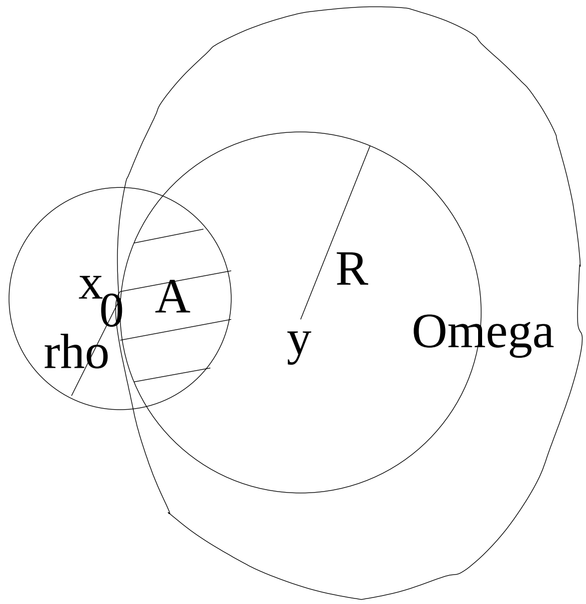

Figure 1: Balls $B_R(\mathbf{y}) \subset \Omega$ and $B_\rho(\mathbf{x}_0)$, with $\mathbf{x}_0 \in \partial B_R(\mathbf{y})$

Lemma 4 *Suppose that Ω is on one side of $\partial\Omega$. Assume $Lu \geq 0$ and let $\mathbf{x}_0 \in \partial\Omega$ be a point such that $u(\mathbf{x}_0) > u(\mathbf{x})$ for every point $x \in \Omega$. Also, assume that $\partial\Omega$ is $\mathbf{C}^2$, at least locally near $\mathbf{x}_0$ and that u is differentiable at $\mathbf{x}_0$. Furthermore assume that either assumptions* **i** *or* **ii** *below hold:*
i. $c = 0$
ii. $\{c \leq 0\ ,\ \text{and } u(\mathbf{x}_0) \geq 0\}$ *or* $u(\mathbf{x}_0) = 0$
Then $\partial_n u(\mathbf{x}_0) > 0$, where ∂_n is the outwards normal derivative.

Proof. We will prove it for the simpler case $Lu := \Delta u + b_i \partial_{x_i} u$ and ask you to complete the proof in the general case by using similar arguments. We construct a ball (see Fig. 1) $B_R(\mathbf{y}) \subset \Omega$ of radius R centered around some point $\mathbf{y} \in \partial\Omega$, with $\mathbf{x}_0 \in \partial B_R(\mathbf{y})$. Also, choose a ball $B_\rho(\mathbf{x}_0)$ of sufficiently small radius ρ centered at $\mathbf{x}_0$. Let $A = B_\rho(\mathbf{x}_0) \cap B_R(\mathbf{y})$. Define $r = |\mathbf{x} - \mathbf{y}|$ and introduce $v(\mathbf{x}) = e^{-\alpha r^2} - e^{-\alpha R^2}$. Note $v = 0$ on $\partial B_R(\mathbf{y})$ and that $1 > v > 0$ in $B_R(\mathbf{y})$. Calculations for the simplified L above shows that

$$Lv = e^{-\alpha r^2} \left(4\alpha^2 r^2 - 2n\alpha - 2\alpha \mathbf{b} \cdot (\mathbf{x} - \mathbf{y})\right)$$

It is clear that for we may choose α sufficiently large so that $Lv \geq 0$ for $\mathbf{x} \in A$. We can then choose sufficiently small $\epsilon > 0$ so that $L\left[u - u(x_0) + \epsilon v\right] \geq 0$ in A, while $u(\mathbf{x}) - u(\mathbf{x}_0) +$

$\epsilon v(\mathbf{x}) \le 0$ on ∂A. From weak maximum principle, $u(\mathbf{x}) - u(\mathbf{x}_0) + \epsilon v(\mathbf{x}) \le 0$ for any $\mathbf{x} \in A$. Now choose $\mathbf{x} = \mathbf{x}_0 - h\mathbf{n}$, where $\mathbf{n}$ is th unit outwards normal to $\partial\Omega$ at $\mathbf{x}_0$. Then noting that $v(\mathbf{x}_0) = 0$, we get for any $h > 0$,

$$\frac{1}{h}\left(u(\mathbf{x}_0 - h\mathbf{n}) - u(\mathbf{x}_0) + \epsilon v(\mathbf{x}_0 - h\mathbf{n}) - \epsilon v(\mathbf{x}_0)\right) \le 0 \tag{9}$$

which in the limit $h \to 0$ gives

$$-\partial_n u(\mathbf{x}_0) - \epsilon \partial_n v(\mathbf{x}_0) \le 0 \tag{10}$$

Since $-\partial_n v(\mathbf{x}_0) = 2\alpha R e^{-\alpha R^2} > 0$, the lemma follows. □

Theorem 5 *(Strong Maximum Principle for Elliptic Equations) Assume $Lu \ge 0$ ($Lu \le 0$ in Ω, which need not be bounded and assume u is not a constant. If $c = 0$, then u does not achieve its maximum (minimum) in the interior of Ω. If $c \le 0$, u cannot achieve a non-negative maximum (non-positive minimum) in the interior. For either sign of c, u cannot be zero at an interior maximum (minimum).*

Proof. We will prove the case for $Lu \ge 0$ since change of sign of u will similarly lead to the case $Lu \le 0$. Suppose u attains a maximum value M at an interior point. Define $\Omega^- = \Omega \cap \{\mathbf{x} : |u(\mathbf{x})| < M\}$. Note Ω^- is non-empty (by non-constant assumption on u) aand $\partial\Omega^- \cap \Omega$ is non-empty because it must contain an interior maximum. Let B be the largest centered at some point $y \in \Omega^-$ entirely contained in Ω^-. Take $\mathbf{x}_0 \in \partial B \cap \partial\Omega^- \setminus \partial\Omega$. Applying previous lemma to B, it follows that $\nabla u(\mathbf{x}_0) \ne 0$, which is impossible for an interior maximum. □

2.1 Application of Maximum Principle: Bounds on solution to elliptic problem $Lu = f$

Definition 6 *Define*

$$\lambda(\mathbf{x}) = \inf_{|\boldsymbol{\xi}|=1} a_{i,j}(\mathbf{x})\xi_i\xi_j \ , \beta = \sup_{\mathbf{x}\in\bar{\Omega}} \frac{|\mathbf{b}(\mathbf{x})|}{\lambda(\mathbf{x})} \tag{11}$$

Theorem 7 *Assume Ω is bounded and contained in the strip between two parallel planes of distance d. Assume that $c \le 0$. If $Lu = f$, then*

$$\sup_{\bar{\Omega}} |u| \le \sup_{\partial\Omega} |u| + C \sup_{\bar{\Omega}} \frac{|f|}{\lambda} \ , \text{where } C = \exp\left[(\beta+1)d\right] - 1 \tag{12}$$

Proof. We shall assume $Lu \geq f$ and prove that

$$\sup_{\bar{\Omega}} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\bar{\Omega}} \frac{|f^-|}{\lambda} \tag{13}$$

The theorem follows once (13) is established by switching $u \to -u$ in the above argument and using $(-u)^+ = -u_-$, $L[-u] = -f$, $(-f)^- = -f^+$, $|f^-| + |f^+| = |f|$ and $|u| = u^+ - u^-$. Also, without loss of generality, we assume $0 < x_1 < d$ since with appropriate translation and rotation of axis, we can arrange any such domain Ω to be bounded by planes $x_1 = 0$ and $x_1 = d$. Note rotation and translation does not affect λ and β. We define operator $L_0 = L - cI$. Then, for $\alpha \geq \beta + 1$,

$$L_0 e^{\alpha x_1} = \left(\alpha^2 a_{11} + \alpha b_1\right) \geq \lambda \left(\alpha^2 - \alpha\beta\right) e^{\alpha x_1} \geq \lambda \tag{14}$$

Define

$$v = \sup_{\partial\Omega} u^+ + [\exp(\alpha d) - \exp(\alpha x_1)] \sup_{\bar{\Omega}} \frac{|f^-|}{\lambda} \tag{15}$$

Then, $v \geq u$ on $\partial\Omega$ and $v \geq 0$ in Ω. Furthermore

$$L(v - u) = L_0 v + cv - Lu \leq -\lambda \frac{|f^-|}{\lambda} - f \leq 0 \tag{16}$$

From the maximum principle, $v \leq u$ in Ω from which (13) follows when we choose $\alpha = \beta + 1$. $\square$

Remark 2 The following Corollary shows that the assumption $c \leq 0$ in Theorem 7 may be relaxed.

Corollary 8 *Let $Lu = f$ in a bounded domain Ω. Let C be as in previous Theorem and assume that*

$$C^* = 1 - C \sup_{\bar{\Omega}} \frac{c^+}{\lambda} > 0 \tag{17}$$

Then

$$\sup_{\bar{\Omega}} |u| \leq \frac{1}{C_*} \left(\sup_{\partial\Omega} |u| + C \sup_{\bar{\Omega}} \frac{|f|}{\lambda} \right) \tag{18}$$

Proof. follows from applying previous theorem to the equation $(L_0 + c^-)u = f - c^+ u$. $\square$