1 Derivatives of a function of a complex variable

Definition 0.1 $f$ has a derivative at $z$ if

$$
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \equiv f'(z)
$$

exists, independent of $\arg h$. Alternatively, $f(z + h) - f(z) = h [f'(z) + \epsilon(z, h)],$ with $\epsilon \to 0$ as $h \to 0.$

Remark: Note continuity of $f$ follows when $f$ is differentiable.

Definition 0.2 $f$ is analytic at $z_0$, if $f'$ exists for $z \in B_\delta(z_0)$ for some $\delta > 0$.

Lemma 0.3 A necessary condition for $f(x + iy) = u(x, y) + iv(x, y)$ to be analytic is that the following Cauchy-Riemann (C-R) conditions are satisfied:

$$
u_x = v_y \ ; \ \ u_y = -v_x$$

Proof If $h$ is real,

$$
f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \left[ \frac{u(x + h, y) - u(x, y)}{h} + i \frac{v(x + h, y) - v(x, y)}{h} \right] = u_x + iv_x
$$

Similar steps for $h$ imaginary shows

$$
f'(z) = v_y - iu_y
$$

Hence C-R conditions are satisfied.

Note: The converse of Lemma 0.3 is not generally true. However, if $u, v \in C^1$, then the converse follows (See O. Costin notes, Thm. 5.8, page 3)

2 Integration, Cauchy’s theorem

Definition We define

$$
\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt \ , \ \int_{-C} f(z)dz = \int_b^a f(\gamma(t))\gamma'(t)dt
$$

where

$$
C = \{ z : z = \gamma(t), a \leq t \leq b \}
$$

Definition A curve $C : \{ z : z = \gamma(t), a \leq t \leq b \}$ is a simple closed curve if $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$, except when $t_1 = a$, $t_2 = b$ for which $\gamma(a) = \gamma(b)$. 


Remark By writing \( f(z) = u + iv, \gamma(t) = \alpha(t) + i\beta(t) \) and separating out real and imaginary parts, we can borrow from real variable theory to find conditions that guarantee existence of \( \int_{C} f(z) \, dz \). The symbol \( f_{-C} \) implies that the integral is along the same curve \( C \), but traversed in opposite direction. From definition, it follows \( f_{-C} = -f_{C} \).

**Theorem 1.1, Cauchy’s theorem**

Let \( f \) be analytic in a region (open connected set) \( R \) in the complex plane \( \mathbb{C} \). Consider any simple piecewise smooth closed curve \( C \), which together with its interior is contained in \( R \) (see Fig. 1). Then

\[
\oint_{C} f(z) \, dz = 0
\]  

**Proof:** Note \( f(z) = u(x, y) + iv(x, y) \), \( dz = dx + idy \); so

\[
\oint_{C} f(z) \, dz = \oint_{C} (udx - vdy) + i \oint_{C} (udy + vdx)
\]

Use of Green’s theorem and CR condition gives the desired result.

**Remark:** Theorem above holds for \( f \) is analytic in \( R \) and continuous in \( \bar{R} \).

![Figure 1: Closed contour C within R](image)

**Corollary 1.2:** If \( f(z) \) is analytic in a simply connected region \( R \). Then for \( z_{0}, z \in R \),

\[
F(z) = \int_{z_{0}}^{z} f(z') \, dz'
\]  

is independent of the contour \( C \) in \( R \), connecting \( z_{0} \) and \( z \), (see Fig. 2). Further, \( F \) is analytic in \( R \) with \( F'(z) = f(z) \).

**Proof:** Consider two contours \( C_{1} \) and \( C_{2} \) (shown in bold and dashed lines respectively in Fig. 2 joining \( z_{0} \) to \( z \) that are entirely within \( R \). It is to be noted that \( C_{1} - C_{2} \) forms a
closed contour that is entirely contained in $R$. Thus from theorem 1.1,

$$0 = \int_{C_1-C_2} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz$$  \hspace{1cm} (3)

and the corollary follows.

**Remark:** Since $F(z)$ is independent of the contour $C$, it can be seen readily that $F(z)$ has a derivative at each $z$ in $R$, since

$$F'(z) = \lim_{\xi \to z} \frac{1}{\xi - z} \int_{\xi}^{z} f(z') \, dz' = f(z)$$  \hspace{1cm} (4)

Hence $F(z)$ is analytic in $R$.

**Remark:** If $G(z)$ is any other function so that $G''(z) = f(z)$, then

$$\frac{d}{dz} (F(z) - G(z)) = 0$$  \hspace{1cm} (5)

This implies, from application of the CR conditions, that $F(z) - G(z) = K$, where $K$ is some complex constant. Actually, since $F(z_0) = 0$, it is seen that $K = -G(z_0)$. Thus

$$\int_{z_0}^{z} f(z') \, dz' = G(z) - G(z_0)$$  \hspace{1cm} (6)

analogous to real functions. Because of this *Fundamental Theorem* all the usual integration formulae for elementary functions, which are analytic, get carried over to complex variables.
Lemma 1.3 Let \( R \) be a simply connected region. Let \( u(x, y) \) be a harmonic function of real variables \( x \) and \( y \), with continuous second derivatives. Then \( u(x, t) = \text{Re} \ F(x + iy) \) for some analytic function \( F(z) \).

**Proof:** Define \( f(x + iy) = u_x(x, y) - iu_y(x, y) \). The Cauchy Riemann conditions for this complex function \( f(z) \) correspond to \( u_{xx} = -u_{yy} \) and \( u_{xy} = u_{yx} \), each of which are satisfied. From given condition on continuity of second derivatives of \( u \), it follows that each of the real and imaginary parts of \( f(z) \) are \( C^1 \) functions of \( x \) and \( y \). Therefore \( f(z) \) is analytic. Define \( G(z) = \int_{z_0}^{z} f(z')dz' \). From previous lemma, \( G(z) \) is analytic with \( G'(z) = f(z) \). Therefore, if the real and imaginary parts of \( G(x + iy) \) are \( g_1(x, y) \) and \( g_2(x, y) \), it follows that

\[
f(x + iy) = u_x(x, y) - iu_y(x, y) \quad G'(x + iy) = g_1(x, y) - ig_2(x, y)
\]

Therefore,

\[
u(x, y) = g_1(x, y) + C
\]

for some constant \( C \) and \( u(x, y) = \text{Re} \ F(x + iy) \) where \( F(z) = G(z) + C \).

**Remark:** From PDE theory, harmonic functions are in \( C^\infty \); so continuity assumptions on second derivatives in Lemma 1.3 is redundant.

Lemma 1.4 Suppose \( R \) is a multiply connected region i.e. not all closed contour in \( R \) contain points exclusively in \( R \). Suppose \( C_1, C_2, C_3, C_4, \ldots, C_n \) are each simple closed curve such that the region inside \( C_1 \) and outside all the inner curves \( C_j \), \( j = 2, ... n \) is entirely within \( R \), including the curves themselves (see Fig. 3). If \( f \) is analytic and single valued in \( R \), then

\[
\sum_{j=1}^{n} \int_{C_j} f(z) \, dz = 0 \tag{7}
\]

where \( C_1 \) is traversed counter-clockwise (positive sense), while all other contours are traversed in the negative sense.

**Proof:** We consider a singly connected domain formed by connecting the inner contours \( C_2, \ldots, C_n \) with the outer contour \( C_1 \) by nearly coincident lines, as shown in Fig. 3 for \( n = 3 \). The contribution \( \int f(z)dz \) from nearly coincident lines vanish when they come close to each other since they are traversed in opposite directions, while \( f \) is continuous. The contour integral for this singly connected domain in the limit of coinciding lines approaches the the left of equation (7). Therefore, from Cauchy’s theorem (Thm. 2.1), (7) follows.

**Remark:** In the special case of \( n = 2 \), if \( C_2 \) is also traversed counter-clockwise like \( C_1 \), then (7) implies:

\[
\oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz \tag{8}
\]

i.e. contours can be shrunk or expanded with no change of the contour integral, provided in the process we do not cross the region \( R \) of analyticity of \( f(z) \).
Lemma 1.5 Let $z$ be a point inside a contour $C$. Then

$$\oint_C \frac{1}{\zeta - z} \, d\zeta = 2\pi i \quad \text{and} \quad \oint_C \frac{1}{(\zeta - z)^n} \, d\zeta = 0 \quad \text{for integer} \quad n \neq 1$$ \hspace{1cm} (9)

where the contour $C$ is understood in the positive (counter-clockwise) sense.

Proof: According to (8),

$$\oint_C \frac{1}{(\zeta - z)^n} \, d\zeta = \oint_{C_\epsilon} \frac{1}{(\zeta - z)^n} \, d\zeta \hspace{1cm} (10)$$

where $C_\epsilon$ is a circle of radius $\epsilon$ in the $\zeta$ plane around $\zeta = z$, traversed in the positive sense. On $C_\epsilon$, $\zeta - z = \epsilon e^{i\phi}$ is a parametrization of the curve. So $d\zeta = i\epsilon e^{i\phi} \, d\phi$. Thus

$$\oint_{C_\epsilon} \frac{1}{(\zeta - z)^n} \, d\zeta = \int_0^{2\pi} i \epsilon^{1-n} e^{i(1-n)\phi} \, d\phi = 0 \quad \text{for} \quad n \neq 1 \quad \text{and} \quad = 2\pi i \text{for} \quad n = 1$$

and (9) follows.

Lemma 1.6 (Cauchy’s integral formula) Let $f(z)$ be analytic in a simply connected region $R$, and $C$ be any closed contour entirely within $R$. Let $z$ be inside $C$ (See Fig. 4). Then

$$\oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = 2\pi i f(z)$$ \hspace{1cm} (11)

Proof: Using lemma 1.5, (1) is equivalent to

$$\oint_C \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = 0$$ \hspace{1cm} (12)

We will now prove (12). For any $\epsilon > 0$, choose $\delta$ so that

$$|f(\zeta) - f(z)| \leq \epsilon$$ \hspace{1cm} (13)
when $\zeta$ is on a circular contour $C_\delta$ of radius $\delta$ around $z$. On $C_\delta$, $\zeta = z + \delta e^{i\phi}$ and $d\zeta = i \delta e^{i\phi} \, d\phi$. Therefore, applying triangular inequality,

$$| \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta | \leq \int_0^{2\pi} d\phi \, \epsilon = 2\pi \, \epsilon$$

and this be made arbitrarily small. Thus the relation (12) is proved since integral $\oint_{C_\delta} = \oint_C$.

**Remark:** If $z$ is outside the contour $C$, the expression on the left of (11) will be zero, because the function $f(\zeta)/(\zeta - z)$ is an analytic function of $\zeta$ on and within the contour $C$.

**Remark:** If $z$ is on the contour $C$, the integral in (11) is not defined in the traditional sense. If $z$ is a point where the contour $C$ has a continuous tangent, then we can draw an approximately semi-circular detour of radius $\epsilon$ so that the closed contour containing this small detour (as shown in Fig. 5) does not contain $z$. In that case, it is clear that

$$\oint_{C_{\text{detour}}} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0$$

But $C_{\text{detour}}$ consists of an open contour $C'$ and a semi-circle $C_{\epsilon}$. On the semi-circle $\zeta - z = \epsilon e^{i\phi}$. So, in the limit $\epsilon \to 0$, using arguments similar to the proof of Lemma 2.5, one can prove that

$$\frac{1}{2\pi i} \int_{C_{\epsilon}} \frac{f(\zeta)}{\zeta - z} \to -\frac{1}{2} \, f(z)$$

On using (15), $\oint_{C'} = -\oint_{C_{\epsilon}}$. Therefore, as $\epsilon \to 0$,

$$\frac{1}{2\pi i} \int_{C'} \frac{f(\zeta)}{\zeta - z} \to \frac{1}{2} \, f(z)$$

Figure 4: $z$ within closed contour $C$
The integral above, in the limit $\epsilon \to 0$ is denoted by the symbol $\int$. From the above, we get

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2} f(z) \quad (18)$$

Figure 5: Deformed contour $C_{\text{detour}}$ consisting of $C'$ and $C_{\epsilon}$

**Remark:** If $z$ is a point on the boundary of $C$, where the boundary does not have a smooth tangent, but instead makes an angle $\theta_0$ (see Fig. 6), then it is possible to make a slight modification of the arguments above to show that

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{\theta_0}{2\pi} f(z) \quad (19)$$

**Exercise:** Prove relation (19).
3 Higher derivatives and application of Cauchy’s theorem

**Lemma 1.7** Let \( f(z) \) be analytic in a simply connected region \( R \), and \( C \) be any closed contour in the positive sense that is entirely within \( R \). Let \( z \) be inside \( C \). Then

\[
\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta = f'(z) \tag{20}
\]

**Proof:** Using (11), it is clear that

\[
\frac{f(\xi) - f(z)}{\xi - z} = \frac{1}{2\pi i} \oint_C \left[ \frac{f(\zeta)}{\zeta - \xi} - \frac{f(\zeta)}{\zeta - z} \right] \frac{d\zeta}{\xi - z} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - \xi)(\zeta - z)}
\]

\[
= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} \left( 1 + \frac{\xi - z}{\zeta - \xi} \right) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} + \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} \left( \frac{\xi - z}{\zeta - \xi} \right) \tag{21}
\]

Now, let \( L \) be the length of the contour, \( d = dist(z, C) \) (see Fig. 4) and \( M = \text{Max} |f(\zeta)| \) on and inside \( C \). Then for any \( \epsilon > 0 \), choose \( \delta = \text{Min} \left\{ \frac{d}{2}, \frac{\pi \epsilon d^3}{3 M L} \right\} \). Then for \( |\xi - z| < \delta \) it is clear that

\[
\left| \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2(\zeta - \xi)} \left( \frac{\xi - z}{\zeta - \xi} \right) \right| \leq \frac{M L \delta}{2 \pi d^3/2} \leq \epsilon \tag{22}
\]

Examining (21), in light of (22), we get

\[
\lim_{\xi-z \to 0} \frac{f(\xi) - f(z)}{\xi - z} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} \tag{23}
\]

Hence (20) follows and the lemma is proved.
Remark: Using similar arguments, with (20), as the starting point, we can write \( f'' \) as a divided difference of the first derivative and prove (similar to above) that

\[
f''(z) = \frac{2}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^3}
\]  

(24)

Through routine inductive procedure, it follows that

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}
\]  

(25)

Remark: An function analytic at \( z \) has derivatives at that point to arbitrary order.

Remark: The Cauchy integral formula (20) and its variants are very useful in obtaining definite integrals that could be related to a closed path integral. We will see many examples of this when we do contour integration exercises.

**Lemma 1.8** (Mean value theorem on a circle): If \( f(\zeta) \) is analytic on and inside a circle of radius \( r \) about \( \zeta = z \), then

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta f(z + re^{i\theta})
\]  

(26)

i.e the average value on a circle equals the functional value at the center.

**Proof:** Choose \( C \) to be a circle of radius \( r \) in the \( \zeta \)-plane around \( \zeta = z \). Then on substituting \( \zeta = z + re^{i\theta}, \ d\zeta = ire^{i\theta} \) back into the Cauchy integral formulae:

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{d\zeta \ f(\zeta)}{\zeta - z}
\]

the conclusion (26) follows.

**Corollary 1.9:** Let \( A \) denotes the interior of a circle of radius \( R \) centered around \( \zeta = z \). Let \( f(\zeta) \) be analytic on \( A \) and its closure, then

\[
f(z) = \frac{1}{\pi R^2} \int_A f(z + re^{i\theta}) \, dA
\]  

(27)

This means that the mean value inside the circle is the same as at its center.

**Proof:** Simply multiply (26) by \( r \) and integrate in \( r \) from 0 to \( R \), and divide the resulting expression by \( R^2/2 \) to obtain (27).

**Theorem 1.10** (Maximum Modulus Theorem): Let \( f(z) \) be analytic inside and on a closed contour \( C \). Then \(|f(z)|\) attains its maximum value \( M \) on \( C \). Further if \( z_0 \) is an interior point where \(|f(z_0)| = M \), then \( f(z) \) is identically a constant.

**Proof:** Suppose, \( z_0 \) is an interior point where \(|f(z_0)| = M \). Choose \( R \) small enough so that a circle of radius \( R \) is entirely within \( C \). Applying triangular inequality to (27), it follows that

\[
M = |f(z_0)| \leq \frac{1}{\pi R^2} \int_A |f(z_0 + r e^{i\theta})| \leq M
\]  

(28)
Now, claim that the equality in (28) can only hold if $|f(z_0 + r e^{i\theta})| = M$ for any $r$ between 0 and $R$. Note that equality in (28) implies

$$\frac{1}{\pi R^2} \int_A [M - |f(z_0 + r e^{i\theta})|] \, dA = 0$$

(29)

The integrand in (29) is non-negative. If it is positive for some $z_0 + r e^{i\theta}$, from continuity, it must be so in a neighborhood of that point. In that case, the integral on the left of (29) must be positive, contradicting the equation. Thus $|f(z_0 + r e^{i\theta})| = M$ for $0 \leq r \leq R$, i.e. everywhere inside a circle of radius $R$. We can now choose a point $z_1 \neq z_0$, where $|f(z_1)| = M$. We take a circle of radius $R_1$ that is entirely contained within $C$ (see Fig. 7), and then establish $|f|$ to be constant. Continuing this procedure, at every point inside $C$ $|f|$ will be a constant. If $M = 0$, then $f$ equals constant 0 everywhere. If $M > 0$, then $Re \ln f = \ln M = \text{Constant}$. Applying the C-R conditions, $Im \ln f = \text{Constant}$. Hence $\ln f$ and therefore $f$ is a constant.

Exercise: Determine maximum value of $\sin z$ in $|z| \leq 1$

Corollary 1.11: If $u(x, y)$ is harmonic on and inside a closed contour $C$, then $u(x, y)$ attains its maximum and minimum on its boundary, unless it is identically a constant.

Proof: We know there exists analytic function $f(z)$ on and within $C$, so that $Re f(x + iy) = u(x, y)$. Now, define $g(z) = e^{f(z)}$. Then from maximum modulus theorem, $g(z)$ attains its maximum on its boundary, unless it is a constant. Thus, $|e^f| = \exp(Re f) = e^u$ attains its maximum on the boundary, i.e. $u(x, y)$ attains the maximum on the boundary.
For proving the minimum, choose \( g(z) = e^{-f(z)} \) and repeat the same argument as above, noting that the maximum of \( e^{-u} \) corresponds to the minimum of \( u \).

**Remark:** Actually the assumption in Corollary 1.11 on \( u(x, y) \) being harmonic on the boundary \( C \) can be relaxed. We only need continuity upto the boundary.

**Theorem 1.12 (Liouville’s theorem):** A bounded analytic function in all of the complex plane must be a constant.

**Proof:** Let \( z \) be an arbitrary point of the complex plane. Take a circle of radius \( R \) around \( \zeta = z \) as the contour \( C \) in the complex \( \zeta \)-plane. Then, on \( C \), \( \zeta = z + R \, e^{i\theta} \), \( d\zeta = i \, R \, e^{i\theta} \, d\theta \).

So, if we substitute these expressions into:

\[
f'(z) = \frac{1}{2\pi i} \oint_C \frac{d\zeta \, f(\zeta)}{(\zeta - z)^2}
\]

we obtain from triangular inequality

\[
|f'(z)| \leq \frac{M}{R}
\]

where \( M \) is a finite upper-bound of \( |f| \) in the complex plane. Taking the limit of \( R \to \infty \), it follows that \( f'(z) = 0 \). This is true for any \( z \), therefore \( f(z) \) must be a constant.

**Remarks:** Liouville’s theorem is very useful in a number of context. For instance, it can be used to prove that a polynomial of degree \( n \) has exactly one \( n \) generally complex roots (including multiplicity). First, it is shown that the polynomial has at least one root. This is done by noting that otherwise \( 1/p(z) \) is a bounded analytic function and therefore a constant. But, for a nontrivial polynomial, this is not the case. Hence \( p(z) \) must have one root.

**Remarks:** Liouville’s theorem is also useful in completely characterizing functions, once their singularities are specified.

**Eg.** Determine the most general form of the single valued \( f \), bounded at \( \infty \) and analytic everywhere except at \( z = z_0 \) where

\[
\lim_{z \to z_0} \left( f(z) - \frac{1}{z - z_0} \right) = c
\]

for some constant \( c \).

**Answer:** Define

\[
g(z) = \left( f(z) - \frac{1}{z - z_0} \right)
\]

From the given conditions \( g(z) \) is a bounded entire (analytic everywhere in \( \mathbb{C} \)) function. From Liouville’s theorem, \( g(z) = \) constant. However, since \( g(z_0) = c \), \( g(z) = c \). Hence \( f(z) = \frac{1}{z - z_0} + c \).