Week 4 notes, Math 7651

1 More examples on contour integration

Exercise 8.1: Compute

$$H(a) \ = \ \int_{-\infty}^{\infty} \frac{e^{-i\ a\ z}}{\sqrt{z+i} \ + \ \sqrt{z+3i}} \ dz \tag{1}$$

where $\sqrt{z+i} = |z+i|^{1/2} \ \exp[i \ arg(z+i)/2]$, $\sqrt{z+3i} = |z+3i|^{1/2} \ \exp[\ i \ arg(z+3i)/2]$, and $arg\ (z+i)$, $arg\ (z+3i)$ are each in $[-\pi/2,\ 3\pi/2)$.

Solution: Clearly, for $a \ < \ 0$, $H(a) \ = \ 0$ since on closing the contour from above, Jordan's Lemma applies. We now consider $a \ > \ 0$. Consider the closed contour C, as shown in Fig. 1.

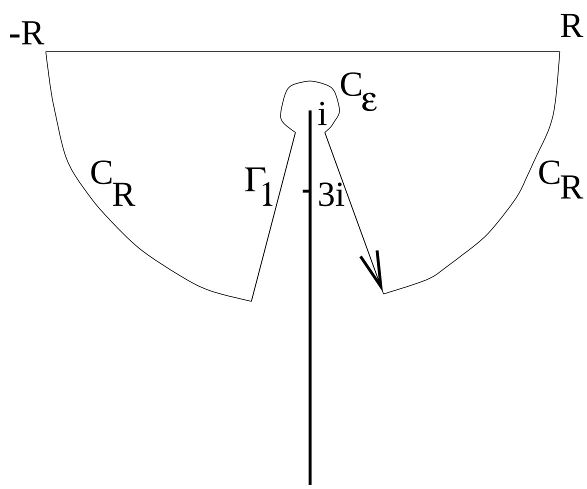

Figure 1: Closed contour C in eqn. 2

Then, since the integrand has no singularity within the contour,

$$\oint_C \frac{e^{-i\ a\ z}}{\sqrt{z+i} \ + \ \sqrt{z+3i}} \ dz \ = \ 0 \tag{2}$$

Using arguments as in Jordan's lemma, there is no contribution from the circular arc portion of the contours, as $R \ \to \ \infty$. We are only left with the contribution from the segment Γ_1 of the contour. For analyzing this contribution, it is prudent to get rid of the squareroot in the integrand in the denominator, by multiplying both the numerator and the denominator by $\sqrt{z+3i} - \sqrt{z\ +\ i}$. Then it is clear from (2) that as $R\ \rightarrow\ \infty$

$$H(a) = \ -\frac{i}{2}\int_{\Gamma_1} e^{-i\ a\ z}\ \sqrt{z+3i}\ dz\ +\ \frac{i}{2}\int_{\Gamma_1} e^{-i\ a\ z}\ \sqrt{z+3i}\ dz \tag{3}$$

Consider

$$H_1(a)\ =\ \int_{\Gamma_1} e^{-i\ a\ z}\ \sqrt{z+3i}\ dz\ =\ \int_{\Gamma_2} e^{-i\ a\ z}\ \sqrt{z+3i}\ dz \tag{4}$$

where Γ_2 is the contour shown in Fig. 1.

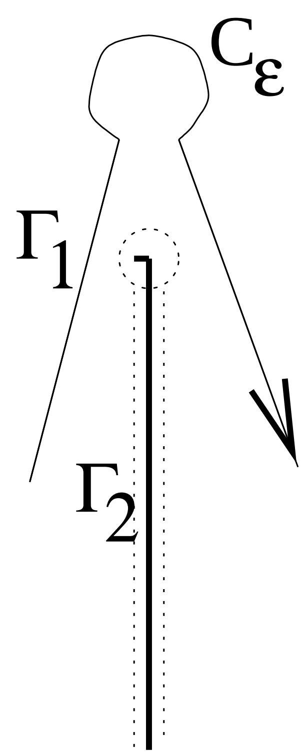

Figure 2: Contour Γ_2, deformed from Γ_1

The latter equality in (4) follows from contour deformation of Γ_1 into Γ_2, as it crosses no singularities of the integrand. The local contribution from the circular arc C_ϵ around "weak" (*i.e.* with power $> \ -1$) branch point $z\ =\ -3\ i$ vanishes as $\epsilon\ \rightarrow\ 0$, following arguments from last lecture. On using $z + 3i \;=\; r\, e^{i3\pi/2}$ on the left leg of Γ_2 and $z \;+\; 3i \;=\; r\, e^{-3i\pi/2}$ on the right leg, we obtain in the limit $\epsilon \;\to\; 0$ and $R \;\to\; \infty$,

$$H_1(a) \;=\; 2\, e^{-\;3\;i\pi/4} \int_0^\infty dr\; r^{1/2}\; e^{-a\;(3\;+\;r)} \;=\; -2\, e^{-3a\;-\;3\;i\pi/4}\, \frac{\partial}{\partial a} \left[\int_0^\infty dr\; r^{-1/2}\; e^{-a\;r} \right]$$
$$= \frac{\sqrt{\pi}}{a^{3/2}}\; e^{-3a\;-\;3\;i\pi/4} \qquad (5)$$

Going through a very similar process it is clear that

$$H_2(a) \;=\; \int_{\Gamma_1} e^{-i\;a\;z}\; \sqrt{z+3i}\; dz\; dz \;=\; = \frac{\sqrt{\pi}}{a^{3/2}}\; e^{-a\;-\;3\;i\pi/4} \qquad (6)$$

From (3), it is clear that

$$H(a) \;=\; \frac{i}{2}\; [H_2(a) \;-\; H_1(a)] \;=\; \frac{\sqrt{\pi}}{2}\; a^{-3/2}\;\; e^{-i\pi/4} \left(e^{-a} - e^{-3a} \right) \qquad (7)$$

Remark: Sometimes, in computing an integral, it is suitable to take derivative with respect to some parameter, as in (5) above, in an effort to simplify the integral. For instance, in calculating

$$I(t) \;=\; \int_{-i\;\infty}^{i\;\infty} \frac{e^{st}}{(s+a)\sqrt{s}}\; ds \qquad (8)$$

we note that $I(t) \;=\; e^{-at}\; I_1(t)$, where

$$I_1(t) \;=\; \int_{-i\;\infty}^{i\;\infty} \frac{e^{(s+a)t}}{(s+a)\sqrt{s}}\; ds \qquad (9)$$

Then, if it is permitted to take derivative with respect to t inside the integral, then we note that

$$I_1'(t) \;=\; \int_{-i\;\infty}^{i\;\infty} \frac{e^{(s+a)t}}{\sqrt{s}}\; ds \;=\; e^{at} \int_{-i\;\infty}^{i\;\infty} \frac{e^{st}}{\sqrt{s}}\; ds \qquad (10)$$

which is a lot easier integral to calculate. In order to find $I_1(t)$ and hence $I(t)$, we integrate the answer in (10). To determine the constant of integration, we note that the integral for $I_1(0)$ in (9) can be calculated easily, through a change in variable ($s_1 \;=\; \sqrt{s}$). However, care must be taken to differentiate under under the integral sign and be certain that this operation is valid.

Exercise 8.2: Evaluate

$$f(z) \;=\; \int_{-1}^{1} \frac{dt}{(t-z)\;\sqrt{1-t^2}} \qquad (11)$$

for z not in $[-1,1]$, and squareroot interpreted in the usual sense of a real positive number.
Solution: Consider

$$\oint_C \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta \tag{12}$$

for a contour C enclosing the cut between -1 and +1, but not enclosing $\zeta = z$ (Fig. 1). Here we choose arg $(\zeta \pm 1) \in (-\pi,\ \pi]$.

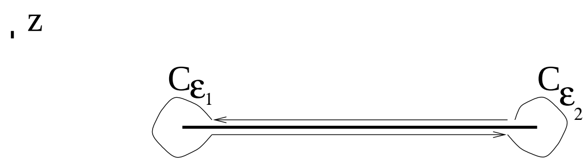

Figure 3: Closed contour C in (12)

Note that since the contribution from the small circular arc contours C_{ϵ_1} and C_{ϵ_2} cannot contribute in the limit as radius $\epsilon \rightarrow 0$, since each is around a weak branch point, with power > -1. Note that on the straight segment of Fig. 1 above the branch cut, arg $(\zeta - 1) = \pi$, where as arg $(\zeta + 1) = 0$, where as below the cut, arg $(\zeta - 1) = -\pi$, arg $(\zeta + 1) = 0$. Therefore, as $\epsilon \rightarrow 0$,

$$\oint_C \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta = -2\ i \int_{+1}^{-1} \frac{1}{(t - z)\ (1 - t^2)^{1/2}}\ dt = 2\ i\ f(z) \tag{13}$$

Now it is clear from Cauchy's integral formulae for a multiply connected (see Fig. 1) region that

$$\left(\oint_{C_R} - \oint_C\right) \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta = 2\pi i \text{ (residue at } \zeta = z)$$

$$= \frac{2\ \pi\ i}{(z^2 - 1)^{1/2}} \tag{14}$$

However, on C_R, as $R \rightarrow \infty$

$$\left| \int \frac{1}{(\zeta - z)\ (\zeta^2 - 1)^{1/2}}\ d\zeta \right| \leq \int_0^{2\pi} \frac{R}{(R - |z|)\ (R^2 - 1)^{1/2}} d\theta \rightarrow 0 \tag{15}$$

Therefore, from (13) and (14), it follows that

$$f(z) = -\frac{\pi}{(z^2 - 1)^{1/2}} \tag{16}$$

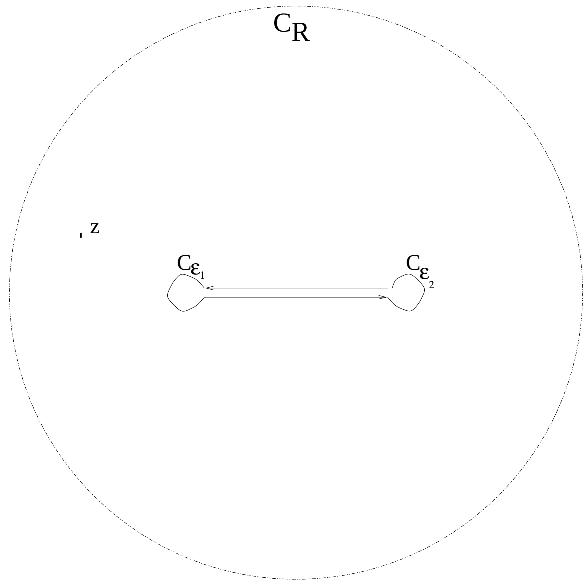

Figure 4: Cauchy integral formulae applied the region between C and C_R

Remark 1 *In other exercises of this type, we may find that the contribution from* $\oint_{C_R}$ *tends to some known constant as* $R \to \infty$*. Even in that case, if there is no branch point at* ∞ *and no cuts going there, it is useful to expand out an initial closed contour* C *around a cut to a large contour* C_R*.*

Remark 2 *For integrals involving periodic function over a period (or something that can be extended to a period), it is useful to relate to a closed complex contour through a change in variable. Here is an example below.*

Exercise 8.3: Compute

$$J = \int_0^{\pi} \frac{d\theta}{1 \, + \, \epsilon \, cos \, \theta} \tag{17}$$

for $-1 < \epsilon < 1$

Solution: Note that since the integrand is even,

$$J = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{1 \, + \, \epsilon \, cos \, \theta} \tag{18}$$

Substituting $z = e^{i\theta}$, $\cos \theta = \frac{1}{2} (z + 1/z)$ and $d\theta = dz/(iz)$. So, the integral (18) becomes

$$J = \frac{1}{2} \oint_{|z|=1} \frac{dz}{iz\left(1 \ + \ \frac{\epsilon}{2}(z + 1/z)\right)} \ = \ \frac{1}{i\epsilon} \oint_{|z|=1} \frac{dz}{z^2 \ + \ \frac{2}{\epsilon} z \ + \ 1} \tag{19}$$

Denoting the two roots of the quadratic in the denominator of the integrand in (19) by z_1 and z_2, it is clear

$$z_{1,2} \ = \ -\frac{1}{\epsilon} \ \pm \ \sqrt{\frac{1}{\epsilon^2} - 1} \tag{20}$$

The product of the roots $z_1 \ z_2 \ = \ 1$ and only z_1 is inside a contour of radius 1. Note that the denominator in the integrand in (19) can also be written as $(z - z_1)(z - z_2)$. So,

$$J \ = \ 2\pi i \ \frac{1}{i\epsilon} \ \lim_{z \ \to \ z_1} \ \frac{(z - z_1)}{(z - z_1) \ (z - z_2)} \ = \ \frac{\pi}{\sqrt{1 - \epsilon^2}}$$

Remark 3 *Contour integation can also be used to compute infinite sum and product representation of meromorphic functions (analytic functions, whose only singularities are poles).*

Example 8.4: Consider

$$P_N(z) \ = \ \frac{1}{2\pi i} \oint_{C_N} \ \frac{\tan \ (\pi \ \zeta) \ d\zeta}{\zeta(\zeta - z)} \tag{21}$$

where C_N is a square contour with corners at $N(\pm 1 \ \pm i)$, where N is a positive integer that will eventually be made to tend to ∞ (See Fig. 1).

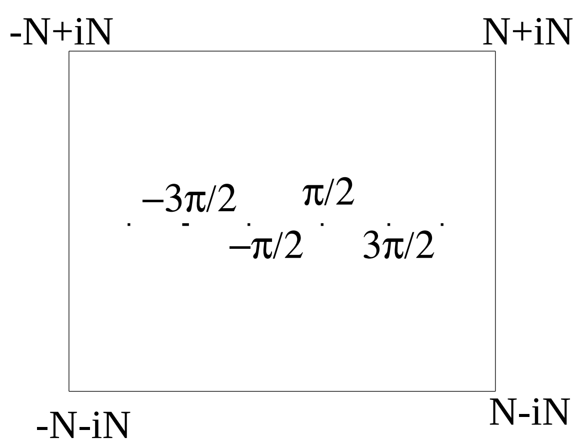

Figure 5: Square contour C_N in eqn. 20

Now, using the exponential representation of sin and cos, it can be easily be shown that $\tan\ [\pi\ (\pm N\ +i\ y)]$ and $\tan\ [\pi\ (x\ +\pm i\ N)]$ are bounded by a constant independent of N. Thus, it is clear from (20) that as $N\ \rightarrow\ \infty$, $P_N\ \rightarrow\ 0$ (leave the details for the reader to fill up). But P_N encloses singularities at $\zeta = z$ and at $\pm\zeta_n$, where $\zeta_n\ =\ (n + 1/2)$ for $N - 1\ \geq\ n\ \geq\ 0$ At $\zeta = z$ the residue is clearly $\frac{tan\ \pi z}{z}$. At $\zeta = \pm\zeta_n$, the residue is clearly $-\frac{1}{\pi(\pm\zeta_n)\ (\pm\zeta_n\ -\ z)}$. Collecting all the residues

$$P_N\ =\ \frac{tan\ \pi z}{z} - \frac{1}{\pi}\sum_{n=0}^{N-1}\left\{\frac{1}{z - \zeta_n}\ \frac{1}{z + \zeta_n}\right\}$$

Taking the limit $N\ \rightarrow\ \infty$, we get

$$\tan\ \pi z\ =\ \frac{2\ z}{\pi}\ \sum_{n=0}^{\infty}\ \frac{1}{\zeta_n^2 - z^2} \tag{22}$$

where $\zeta_n = (n + 1/2)\ \pi$. Noting that $\frac{d}{dz}\log\cos(\pi z) = -\pi\tan(\pi z)$, we can use integration to determine infinite product representation for $\cos(\pi z)$.

2 Analytic homeomorphism between domains: introduction

Remark: Please check O. Costin's Complex Variable notes, pages 34-67. The treatment here is similar, though not as extensive.
Remark: Laplace equation is common in the physical sciences, electro-statics, potential flow in fluid mechanics, steady-state heat diffusion problems to name a few:

$$\Delta \phi \equiv \phi_{xx} + \phi_{yy} = 0 \tag{23}$$

for $x + iy \in D \subset \mathbf{C}$, is a simply connected region with either *Dirichlet* or *Neumann* boundary conditions on $\Gamma \equiv \partial D$:

$$\phi = g(t) \quad \text{(Dirichlet)}, \quad \text{or} \quad \frac{\partial \phi}{\partial n} = g(t) \quad \text{(Neumann)} \tag{24}$$

where $a \le t \le b$ parametrizes Γ and $\frac{\partial}{\partial n} \equiv \mathbf{n} \cdot \nabla$ denote normal derivative on Γ, and g is a known function

When D is not finite, additional conditions on ϕ have to be appended at ∞. In electro-statics for instance, electric field $\mathbf{E} = -\nabla\phi$ In a region without charges, $\nabla \cdot \mathbf{E} = 0$; so $\Delta\phi = 0$.

Remark 4 *Uniqueness of solution to* (23)-(24). *For Dirichlet problem, we already know from application of maximum modulus theorem, that for finite domain ϕ attains a maximum or minium on Γ. The same can be derived for infinite domain D with some mild condition at ∞ by using Phragmen-Lindeloff principle. Therefore, uniqueness of solution to* (23) *for Dirichlet boundary condition in* (24) *assuming solution to be continuous upto the boundary follows. Similar arguments using conjugate harmonic function show that the solution to the Neumann problem is unique upto an additive constant iff $\int_a^b g(t)dt = 0$, when t is the arc-length parametrization.*

2.1 Existence of solution to (23)-(24)

When $D = D_1$, the unit circle, then Poisson-integral formula explicitly provides solution to the boundary value problem (1)-(2). In an effort to find solution to (1)-(2) for more general domains D, we seek to find an analytic 1-1 mapping (analytic homeomorphism) $f : D_1 \to D$ since Laplace's equation is invariant under such mapping as proved in the following Lemma.

Lemma 1 *If ϕ is harmonic in (x, y) for $z = x + iy \in D$, a simply connected domain, and $h : D \to D_1$ is an analytic homeomorphism, then for $z = f^{-1}(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$, where $\zeta = \xi + i\eta \in D_1$,*

$$\Phi(\xi, \eta) = \phi(u(\xi, \eta), v(\xi, \eta))$$

is harmonic in (ξ, η).

Proof. Define $g(z) = g(x + iy) = \phi(x, y) + i\psi(x, y)$, where g is analytic in D. Since $x + iy = z = f^{-1}(\zeta) = f^{-1}(\xi + i\eta)$, it follows that

$$\Phi(\xi, \eta) = \Re\left\{g\left(f^{-1}(\zeta)\right)\right\}$$

Since composition of analytic functions is analytic, $g(f^{-1}(\zeta))$ is an analytic function of ζ, implying Φ to be harmonic in (ξ, η). $\square$

Remark 5 *We now turn to the question of existence of such analytic homeomorphism. This is guaranteed by the Riemann mapping theorem, as stated below for simply connected domains. We will remark on the proof of this theorem once we see get familiar on its application.*

Theorem 2 *(Riemann Mapping Theorem) Assume D is a simply connected open domain with more than one boundary point. Then there exists an analytic homeomorphism $f : D \to D_1$. This map is unique if some $z = z_0 \in D$ is required to to correspond to $0 \in D_1$ with $f'(z_0)$ real and positive.*

Remark 6 *There are similar theorems for mappings between multiply connected regions with same connectivity, though with additional restrictions. Two such domains are called conformally equivalent if there is an analytic homeomorphism between the two.*

Exercise 9.1: Let Γ_1 and Γ_2 be the half-lines arg $z = 0$ and arg $z = \frac{3\pi}{2}$, respectively, and let D be the domain $0 < arg\ z < \frac{3\pi}{2}$ (See Fig. 2.1. Determine a bounded solution $\phi(x, y)$ to Laplace's equation $\Delta\ \phi = 0$, subject to the conditions $\phi = a$ on Γ_1 and $\phi = a + k$ on Γ_2, where a and k are real constants.
Solution: It is to be noted that if we apply the transformation

$$\zeta = h(z) = \frac{2}{3\pi}\ ln\ z \text{ with } 0 < arg\ z < \frac{3\pi}{2} \tag{25}$$

to the domain D, then the corresponding image in the ζ plane corresponds to the region D' between two parallel straight lines $Im\ \zeta = 0$ and $Im\ \zeta = 1$ (see Fig. 2.1). It is to be noted that the inverse of the transformation in (25) is given by

$$z = f(\zeta) = \exp\left[\frac{3\pi}{2}\ \zeta\right] \tag{26}$$

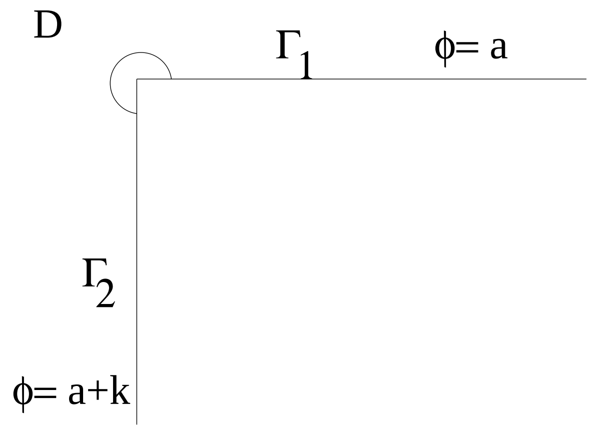

Figure 6: Boundary value problem in D, bounded by Γ_1 and Γ_2

The given boundary conditions in D translate to

$$\Phi \;=\; a \;\text{ on } \eta = 0 \;\;,\;\; \text{ and } \Phi \;=\; a \;+\; k \text{ on } \eta = 1 \tag{27}$$

In the domain D', a solution $\Phi(\xi, \eta)$ is sought in the form $\Phi(\eta)$. In that case, Laplace's equation reduces to

$$\Phi_{\eta\eta} \;=\; 0 \tag{28}$$

The general solution to (28) is in the form $c_1 \;+\; c_2\,\eta$. Since it must satisfy the boundary conditions (27), one finds

$$\Phi \;=\; a \;+\; k\,\eta \;=\; Re\,(-i\;k\;(\xi \;+\; i\eta)\;+\;a) \tag{29}$$

This means that at least one solution for the complex potential is

$$\chi(\zeta) \;=\; a \;-\; i\;k\;\zeta \tag{30}$$

Therefore

$$\Omega(z) \;=\; \chi(h(z)) \;=\; a \;-\; \frac{2ik}{3\pi}\;\log\;z \;\;, \tag{31}$$

implying that

$$\phi \;=\; \frac{2k}{3\pi}\;\arg\;z \;+\; a \;\;\text{with } 0 \;<\; \arg\;z \;<\; \frac{3\pi}{2} \tag{32}$$

The uniqueness can be proved by application of Phragmen-Lindeloff principle under rather weak assumptions at ∞ (This will be an exercise).

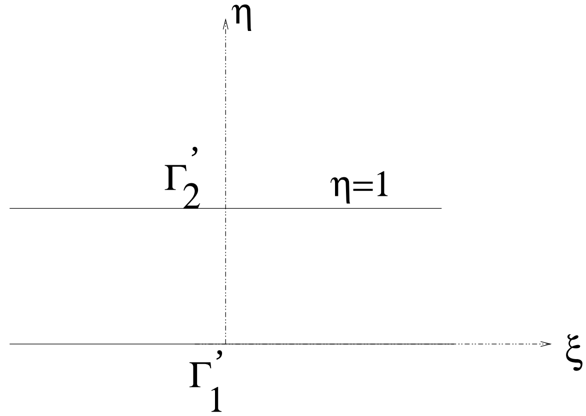

Figure 7: Domain D', bounded by Γ'_1 and Γ'_2, in the $\zeta = \xi + i\eta$ plane

Remark 7 *Some assumption is clearly neede for uniqueness of ϕ, since we can add to* (31) *any solution $\Lambda_j(z) \;=\; i\; z^{2j/3}$ for any nonzero integer j and yet given satisfy boundary conditions on $\phi \;=\; \Re\, \Omega$.*

Remark 8 *A problem of considerable interest in potential theory to detemine Green's function $g(x, y; x_0, y_0)$ for a simply connected domain D. In 2-D, it is defined to be a single-valued harmonic function of (x, y), except only at (x_0, y_0), where*

$$g(x, y; x_0, y_0) - \frac{1}{2\pi} \ln\; |z - z_0| \;=\; O(1) \tag{33}$$

where $z = x \;+\; iy$ and $z_0 \;=\; x_0 \;+\; i\; y_0$. For Dirichlet problem, we require $g \;=\; 0$ on the boundary $\Gamma = \partial D$. From Riemann mapping theorem, there exists $f : D_1 \to D$ (D_1 unit disk around the origin) with $f(0) = z_0$; this mapping is unique upto a rotational degree inherent in choosing $\arg f'(0)$. *We denote the inverse mapping $h(z; z_0) = f^{-1}(z)$, the additional parameter z_0 in h indicates dependence of mapping f on $z_0 = f(0)$. Since* $h(z) = h'(z_0)(z - z_0) + \text{regular terms}$, *it follows* $\Re \log h = \log |z - z_0| + \text{harmonic}$. *Further, since h is 1-1 and has no other zeros in D, it follows that* $\log h$ *is analytic elsewhere and so* $\Re \log h(x + iy; z_0)$ *is harmnonic in D and takes 0 value on ∂D, $|h(z)| = 1$. Hence*

$$g(x, y; x_0, y_0) = \frac{1}{2\pi} \log |h(x + iy; x_0 + iy_0)| \tag{34}$$

3 Conformal transformations:

Remark: The examples from last time illustrate the need to find an analytic function $f(\zeta)$ that maps some standard domain D' into the region D in a one-one manner. We now discuss such mapping functions and their properties without reference to Laplace's equation.
Definition: A transformation $f : D' \to D$ is said to be *conformal* at a point $\zeta_0 \in D'$ if the angle of intersection between any two smooth curves at that point as well as their relative orientation is preserved. The transformation is conformal in D', then this property is valid at every point in D' (See Fig. 3).

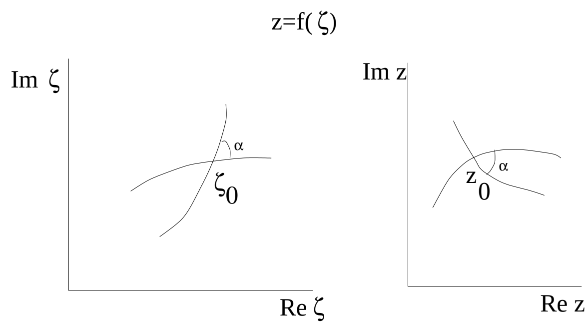

Figure 8: Preservation of angle α and orientation, under transformation $z = f(\zeta)$

Lemma 3 *A function f analytic at ζ_0 is conformal at ζ_0 if $f'(\zeta_0) \neq 0$.*

Proof. Consider two differentiable curves $\Gamma(t)$ and $\gamma(s)$ intersecting at $s = 0, t = 0$, such that $\zeta_0 = \Gamma(0) = \gamma(0)$. The angle between the mapped curves $f(\Gamma(t))$, $f(\gamma(s))$ at $z_0 = f(\zeta_0)$ is clearly

$$\arg\left[\frac{d}{dt}f(\Gamma(t)\right]_{t=0} - \arg\left[\frac{d}{ds}f(\gamma(s)\right]_{s=0} = \arg\left[f'(\zeta_0)\Gamma'(0)\right] - \arg\left[f'(\zeta_0)\gamma'(0)\right] = \arg\left[\Gamma'(0)\right] - \arg\left[\gamma'(0)\right]$$

Therefore the angle and orientation are both preserved $\square$

Remark 9 *The converse is also true. If a map f on a domain $D \in \mathbf{C}$ is conformal and orientation preserving, then it can be proved that f is analytic with $f' \neq 0$. (See Nehari)*

Remark 10 *Consider curve $\Gamma(t)$ in the last Lemma for $a \leq t \leq b$. The infinitesimal arclength change on the transformed curve $f(\Gamma)$ is clearly $|df(\Gamma(t))| = |f'(\Gamma)||\Gamma'|dt$. Therefore arclength is of transformed curve $f(\Gamma)$ is*

$$\int_a^b |f'((\Gamma(t))||\Gamma'(t)|dt$$

So, small length elements at ζ_0 are changed by factor of $|f'(\zeta_0)$. Using Cauchy Riemann conditions, it is clear that if

$$z = x + iy = f(\zeta) = f(\xi + i\eta) = x(\xi, \eta) + iy(\xi, \eta),$$

then area element $d\xi d\eta$ in the ζ plane is transformed to

$$|f'(\zeta_0)|^2 d\xi d\eta = |\frac{\partial\ (x, y)}{\partial\ (\xi, \eta)}|d\xi d\eta \tag{35}$$

Remark 11 *Note that from Taylor expansion that if f is analytic at ζ_0, but $f'(\zeta_0) = 0$, then from Taylor expansion*

$$\delta z = \frac{f^{(m)}(\zeta_0)}{m!}\ (\delta\zeta)^m\ (1 + O(\delta\zeta))$$

where $f^{(m)}(\zeta_0) \neq 0$ is the first nonvanishing derivative. Further, as $\delta\zeta \to 0$,

$$\arg\ \delta z = m\ arg\ \delta\zeta + arg\ f^{(m)}(\zeta_0) + \arg\left(1 + O(\delta\zeta)\right) \tag{36}$$

Thus, angle between any two infinitesimal line elements at the point ζ_0 is increased by the factor m, and therefore the transformation is not conformal at ζ_0.

Remark 12 *Thus, from Lemma 3 and the above remark, a transformation $z = f(\zeta)$ for an analytic function f is conformal at ζ_0 if and only if $f'(\zeta_0) \neq 0$.*

Remark 13 *In an earlier lecture, we noted that for an analytic function f, $f'(\zeta_0) \neq 0$ implies that f is locally invertible and that the inverse function is analytic at $z_0 = f(\zeta_0)$. However, the condition $f' \neq 0$ at each point of D' does not necessarily mean that f is a one-one map i.e. f is an analytic homeomorphism between D and D'. For instance, $f(\zeta) = e^\zeta$ is locally invertible (since $f' \neq 0$); yet if we take D' to be the region $\{\zeta \mid 0 \leq Im\ \zeta < 4\pi\}$, then there exists more than one point in D' (actually, exactly two) with the same image in D.*

Lemma 4 *Assume $f : D' \to D$ is an analytic homeomorphism. Then f and f^{-1} is conformal at each interior point.*

PROOF. Assume other wise; *i.e.* that there exists point $\zeta_0 \in D'$ such that $f'(\zeta_0) = 0$. Denote $z_0 = f(\zeta_0)$. Let m be the smallest integer with $f^{(m)}(z_0) \neq 0$. Since the zeros of a nontrivial f are isolated, there exists ϵ such that or $0 < |\zeta - \zeta_0| < \epsilon$, $f(\zeta) - z_0 \neq 0$. Choose a contour C around ζ_0 and define

$$\delta = \min_C |f(\zeta) - z_0|$$

Now for $|z - z_0| < \delta$, it is clear from Rouche's theorem that the functions $f(\zeta) - z_0$ and $f(\zeta) - z_0 + (z_0 - z)$ have exactly the same number of zeros, which is m (counting multiplicity) > 1. This is a contradiction since f is given to be an analytic homemorphism. So the mapping is conformal at all interior points of D'. The same arguments can be made for $f^{-1} : D \to D'$ since it is also an analytic homeomorphism. □

Remark: Note in the above proof that if $f'(\zeta_0) = 0$, there are many ($m > 1$) branches of an inverse in a neighborhood of $z = z_0$; i.e. $z = z_0$ is a branch point of an inverse function of f.

Lemma 5 *Assume C is a piecewise smooth oriented simple closed curve enclosing a simply connected domain D and f analytic in D and continuous in $\bar{D}$. If $C' = f(C)$ is also a simple closed curve, then f is an analytic homeomorphism between D and $f(D)$. and preserves its orientation.*

PROOF. We parametrize the closed path C by a real parameter $t \in [0, 1]$. and assume it is traversed positively (counter-clockwise) as t increases. Since C is a simple closed curve, this implies that for $t_1 < t_2$, $\zeta(t_1) \neq \zeta(t_2)$ except when $t_1 = 0$ and $t_2 = 1$. From given condition, $C' = f(C)$ is is a simple closed curve as well; or otherwise at least two distinct points on C would map to the same point on C'. Let C' contain the domain D'. Take a point $w_0 \in D'$. We will show that $f(\zeta)$ attains the value w_0 exactly once in D. We note

$$\frac{1}{2\pi i} \oint_C \frac{f'(\zeta) d\zeta}{f(\zeta) - w_0} = \frac{1}{2\pi i} \int_0^1 \frac{f'(\zeta(t))\zeta'(t)\, dt}{f(\zeta(t)) - w_0} \tag{37}$$

Now, since $w(t) = f(\zeta(t))$ is the parametrization of the simple closed C', it is clear that (37) equals

$$\frac{1}{2\pi i} \oint_{C'} \frac{dw}{w - w_0} = \pm 1 \ , \tag{38}$$

depending on the whether C' is oriented counter-clockwise or clockwise. However since expression (37) is the number of times f attains w_0, which must be non-negative, and is equal to the expression (38), it must equal +1. This implies (i) f is an analytic homeomorphism in D and (ii) it preserves the orientation of C. $\square$

Remark 14 *An analytic homeomorphism that preserves its orientation is generally referred to as a conformal mapping, though confusion may arise on whether the mapping is only locally conformal.*

Remark 15 *Because of Lemma 5, in order to determine if an analytic f is conformal and defines a 1-1 map between D and $f(D)$, where D is simply connected with piecewise ∂D, it is enough to check that f is one to one between ∂D and $f(\partial D)$.*

Remark 16 *If the region of interest in the D plane is the exterior of some closed simple curve C on which the analytic map f is one to one, then it is helpful to introduce an intermediate transformation g: $g(\zeta) = \frac{1}{\zeta - \zeta_0}$, where ζ_0 is inside C. g is seen to map the region exterior of C into a domain interior of $g(C)$ in a 1-1 manner. Lemma 5 can then be applied to $f \cdot g^{-1}$ which is a mapping between finite domains $g(D)$ and $f(D)$.*

Remark 17 *If the region of interest is one side of an infinite non-self-intersection curve C in the ζ plane, which is piecwise smooth, we can once again use the transformation g, defined by $g(\zeta) = \frac{1}{\zeta - \zeta_0}$ for ζ_0 on the other side of C, before applying Lemma 5. On transformation, the infinite curve C can be applied to the resulting domain in the $\zeta_1 = g(\zeta)$ domain. This has the effect of transforming an infinite contour into a finite one and mapping one side of it into the interior/exterior of that curve.*

Remark 18 *In the case, f has a simple pole in a finite domain D, we consider first the mapping properties of g defined by $g(\zeta) = \frac{1}{f(\zeta) - a}$, where $a \neq f(\zeta)$ for $\zeta \in \bar{D}$. Since g is analytic and free of singularities, we can use Lemma 5 when applicable. Once mapping properties of g is determined, the explicit relation between f and g allows determination of the mapping properties of f as well.*

3.1 Mapping properties of simple functions:

Example 1: Consider linear mapping $f(\zeta) = a\,\zeta + c$; this corresponds to a dilation of $|a|$ and counter-clockwise rotation by $\arg a$, followed by translation by c.
Example 2: Consider $w = f(\zeta) = \frac{1}{\zeta}$. It maps the exterior of unit circle centered at the the origin into its interior, with ∞ mapped to the origin. More generally, it maps any domain exterior to a curve C that contains the origin into a finite closed domain. It also maps a finite region containing the origin into an infinite region, exterior to the curve $f(C)$.

Generally, it maps circles (straightlines) into circles (or straightlines). To see this, suppose $\zeta = \xi \; + \; i\eta$; then

$$u \; + \; iv \; = \; f(\zeta) \; = \; \frac{1}{\xi \; + \; i\eta}$$

Thus,

$$\xi \; = \; \frac{u}{u^2 \; + \; v^2} \quad , \quad \eta \; = \; - \; \frac{v}{u^2 \; + \; v^2} \tag{39}$$

Then, if

$$\xi^2 \; + \; \eta^2 + A\,\xi \; + \; B\,\eta \; = \; C$$

then from (39)

$$\frac{u^2}{(u^2 \; + \; v^2)^2} \; + \; \frac{v^2}{(u^2 \; + \; v^2)^2} \; + \; \frac{Au}{u^2 \; + \; v^2} \; - \; \frac{Bv}{u^2 \; + \; v^2} \; = \; C$$

which simplifies to

$$1 \; + \; Au \; - B\,v \; = \; C(u^2 \; + \; v^2)$$

If $C = 0$, this is a straight line in the w plane; otherwise it is a circle. Similarly, you can easily show that the map of a straight line in the ζ plane is a circle (or straight line) in the w plane.

Example 3: A fractional linear map (or Mobius map) is defined by

$$w = f(z) = \frac{az + b}{cz + d} \tag{40}$$

where $a/c \; \neq \; b/d$ (otherwise w is a constant). This can be generally viewed as a composition of a linear mapping, an inversion, followed by a linear mapping:

$$w_1 = cz + d \;\; ; \quad w_2 = \frac{1}{w_1} \;\; ; \;\; w \; = \; \left(b - \frac{ad}{c}\right) \; w_2 \; + \; \frac{a}{c}$$

If $c = 0$, then the fractional linear mapping simply reduces to a linear mapping. It is to be noted that the inverse of (40) is once again a fractional linear transformation (as can be checked):

$$z = h(w) = \frac{dw - b}{-cw + a}$$

Fractional linear transformation maps a circle (and straightlines) into a circle or straightlines. It also has the property that if w_1, w_2, w_3 and w_4 are four points corresponding to z_1, z_2, z_3 and z_4, then the cross ratios are equal, *i.e.*

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)} \; = \; \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

It is to be noted that there are only three independent parameters in (40). For instance, for nonzero c, these independent parameters can be take to be a/c, b/c and d/c. These three parameters are uniquely set when the images of any three distinct points z_1, z_2 and z_3 are specified to be the distinct values w_1, w_2 and w_3. The image w_4 of a fourth point z_4 is immediately determined by the cross-ratio relation above.

Remark 19 *Composition of Mobius map is also a Mobius map as is readily checked. Further, since inversion of a Mobius map is again a Mobius map and the set of mappings include the identity mapping, the set of Mobius maps form a group. Indeed, it is easily proved that any analytic homeomorphism between circles is necessarily a Mobius map. This map be used to prove the uniqueness claim of the Riemann-mapping theorem in the following manner. Consider a particular $f : D \to D_1$ (D_1 is the unit disk), whose existence is given by Riemann mapping theorem. Now, the most general map from D_1 to D_1 is a Mobius map g. Therefore, the most general map $F : D \to D_1$ must have the form $F = g \cdot f$. By demanding $F(z_0) = w_0$ and $F'(z_0) > 0$, the Mobius map g is uniquely determined.*