

## 1 More on Conformal mapping

**Theorem 1** ((Schwartz Lemma). Let  $D_1$  be the unit disk. Let  $f : D_1 \rightarrow D_1$  be analytic with  $f(0) = 0$  and continuous in its closure  $\bar{D}_1$ . Then **i.**  $|f(z)| \leq |z|$  for  $z \in D_1$ , **ii.** If  $|f(z_0)| = |z_0|$  for some  $z_0 \in D_1$ , then  $f(z) = e^{i\phi}z$  for some real  $\phi$ . Further, **iii.**  $|f'(0)| \leq 1$ ; **iv.** if  $|f'(0)| = 1$ , then  $f(z) = e^{i\phi}z$ .

PROOF. Note  $g(z) = \frac{f(z)}{z}$ . Note  $g$  is analytic in  $D$ . From maximum modulus Theorem,  $|g(z)| \leq 1$ . So,  $|f(z)| \leq |z|$  as claimed in part **i.** If equality holds for  $z = z_0$ , then from maximum modulus theorem,  $|g(z)| = 1$  everywhere, implying part **ii.** Since

$$|g(h)| = \left| \frac{f(h) - f(0)}{h} \right| \leq 1$$

On taking the limit of  $h \rightarrow 0$ ,  $|f'(0)| \leq 1$ , implying part **iii.** If  $|f'(0)| = 1$ , then  $f'(0) = e^{i\phi}$ . Contrary to what needs to be proved, assume  $f(z) \neq e^{i\phi}z$ . Then, we may decompose

$$\frac{f(z)}{z} = e^{i\phi} (1 + z^m e^{i\psi} q(z)),$$

for some integer  $m \geq 1$ , with  $q$  analytic and  $q(0) > 0$ . Choose  $z_0 = \epsilon e^{-i\psi/m}$  for  $\epsilon > 0$  small. Then, it is clear from continuity of  $q$  that  $|f(z_0)/z_0| > 1$ , which contradicts **i.**  $\square$

**Corollary 2** If  $f : D_1 \rightarrow D_1$  is an analytic homeomorphism with  $f(0) = 0$ , then  $f(z) = e^{i\phi}z$  for some real constant  $\phi$ .

PROOF. From Schwartz Lemma that  $|f(z)| \leq |z|$ . But  $f^{-1} : D_1 \rightarrow D_1$ . and therefore  $|f^{-1}(w)| \leq |w|$ . Therefore,  $|z| \leq |f(z)|$ . Hence  $|f(z)| = |z|$ , implying from Schwartz Lemma that  $f(z) = e^{i\phi}z$ .  $\square$

**Lemma 3** The most general analytic homeomorphism  $f : D_1 \rightarrow D_1$  is given by

$$f(z) = S(z) = e^{i\phi} \left( \frac{\alpha - z}{1 - \bar{\alpha}z} \right), \text{ where } \alpha = f^{-1}(0)$$

PROOF. First, we note on substituting  $z = e^{i\theta}$ ,  $\alpha = \rho e^{i\psi}$ ,  $\rho < 1$ , we obtain

$$|S(e^{i\theta})| = \left| e^{i\phi} \left( \frac{\rho e^{i\psi} - e^{i\theta}}{1 - \rho e^{-i\psi} e^{i\theta}} \right) \right| = 1$$

We then notice  $h(\zeta) = f(S^{-1}(\zeta))$  maps  $D_1$  into itself with  $h(0) = 0$ , and hence from Corollary 2,  $h(\zeta) = e^{i\phi'} \zeta$  implying  $f^{-1}(e^{-i\phi'} \zeta) = S^{-1}(\zeta)$ , i.e.,  $e^{i\phi'} f(z) = S(z)$ . Since  $\phi$  is a general choice, there is no loss of generality in choosing  $\phi' = 0$ , implying  $f(z) = S(z)$ .  $\square$

**Lemma 4** *The most general analytic homeomorphism between simply connected domains  $D$  and  $D'$  each bounded by a circle or a straightline is a Mobius (fractional linear) map.*

PROOF. We already know Mobius transformation maps circles/straight lines into circles/straight lines. Therefore, using appropriate mobius maps  $S_1, S_3$ , each of  $S_1(D), S_3^{-1}(D')$  are unit circles. Using Lemma 3, there exists mobius map  $S_2$  so that  $S_3^{-1}(D') = S_2(S_1(D))$ , implying  $S_3 \circ S_2 \circ S_1 : D \rightarrow D'$  is the analytic homeomorphism between  $D$  and  $D'$ . Since mobius map forms a group, the lemma is proved.  $\square$

## 2 More on properties of simple maps

**Example 4:**  $w = f(z) = z^2$  conformally maps the upper-half plane into the entire cut  $w$  plane, with the cut along the positive real  $w$ -axis. It also maps the region inside the semi-circle into the cut unit-circle; maps a quarter circle into a semi-circle. More generally  $f(z) = z^n$  conformally maps the sector  $0 < \arg < \frac{2\pi}{n}$  into the entire cut  $w$  plane, with cut along the positive real  $w$  axis.

**Example 5:** The exponential function  $w = f(z) = e^z$  maps any strip parallel to the  $Re z$  axis of width  $2\pi$  into the entire  $w$  plane. It's inverse is the  $\log w$  function with cut location dependent on the location of the strip in the  $z$ -plane. So, for a strip, lying between  $\theta_0 < Im z_0 < 2\pi + \theta_0$ , the branch cut in the  $w$  plane has to coincide with  $\arg w = \theta_0$ . with  $\log w$  interpreted as  $\log |w| + i \arg w$ ,  $\arg w$  in  $(\theta_0, 2\pi + \theta_0)$ . Notice that  $e^z$  maps a strip of width  $\pi$  into a half-plane. Also,  $f(z)$  maps the half-strip  $x > 0, -\pi/2 < y < \pi/2$  onto the portion of the right half  $w$  plane that lies entirely outside the unit circle.

**Example 6:**  $z = f(\zeta) = \sin \frac{\pi}{2} \zeta$  conformally maps the half-strip  $-1 < Re \zeta < 1, Im \zeta > 0$  to the upper-half  $z$  plane. This is because for  $\zeta$  on the real axis, in  $(-1, 1)$ , the image is also in  $(-1, 1)$ . For  $\zeta = \pi/2 + i\eta$ , the image is the real  $z$  axis in  $(1, \infty)$  since

$$\frac{1}{2i} [e^{i\pi/2-\eta} - e^{-i\pi/2+\eta}] = \cosh \eta$$

Similarly, the image of  $\zeta = -\pi/2 + i\eta$  is the real  $z$  axis in  $(-\infty, 1)$ . From the orientation of the points mapped on the boundary it is clear, that the mapping is to the upper half  $z$  plane rather than the lower-half plane.

**Example 7:** Joukowski's transformation

$$z = f(\zeta) = \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right)$$

conformally maps the region exterior to  $|\zeta| > 1$  into the region exterior of the straightline slit connecting  $z = -1$  to  $z = 1$ . Note that  $f'(\zeta) = 0$  iff  $\zeta = \pm 1$ . Hence the mapping is not conformal at those boundary points, though it is in the interior. Note, however, it maps circles  $|\zeta| = a > 1$  into ellipses since

$$z = x + iy = \frac{1}{2} \left( \left( a + \frac{1}{a} \right) \cos \theta + \left( a - \frac{1}{a} \right) \sin \theta \right)$$

Therefore,  $f(\zeta)$  maps the exterior of circle  $|\zeta| = a > 1$  into the exterior of an appropriate ellipse in the  $z$ -plane. The inverse of the mapping from the exterior of a circle to the exterior of an ellipse (or a slit) is given by

$$\zeta = h(z) = z + (z^2 - 1)^{1/2}$$

with cuts chosen to coincide with the slit and branch chosen so that  $(z^2 - 1)^{1/2} \rightarrow z$ , as  $z \rightarrow \infty$ . Note that Joukowski's map also maps the interior of a unit semi-circle in the upper-half  $\zeta$  plane into the lower-half  $z$  plane.

**Remark:** It is to be noted that the Joukowski's transformation also maps the interior of the unit circle in the same way because the mapping has the property  $f(1/\zeta) = f(\zeta)$ . The inverse in this case is the other branch

$$\zeta = h(z) = z - (z^2 - 1)^{1/2}$$

In this case, it is to be noted that  $\zeta \rightarrow 0$  as  $z \rightarrow \infty$ , as it should be.

**Remark:** The examples above can be combined to solve quite nontrivial problems of potential theory.

**Exercise:** Find a solution to  $\Delta\phi = 0$  inside a unit quarter-circle in the first quadrant so that  $\phi = 0$  on straight segments and  $\phi = 1$  on the circular arc.

**Solution:** Let  $\phi(x, y) = \text{Re } \Omega(z)$ ,  $z = x + iy$ . Introduce

$$z_1 = f_1(z) = z^2 \tag{1}$$

This maps the quarter circle into the interior of the upper-half unit-semi-circle.  $\phi = 0$  on the real diameter, while  $\phi = 1$  on the circular arc. Now introduce

$$z_2 = f_2(z_1) = -\frac{1}{2}(z_1 + 1/z_1) \quad (2)$$

This maps the inside of the semi-circle into the upper half-plane. Note in the  $z_2$  plane,  $\phi = 1$  in  $(-1, 1)$  and equals 0 on the real axis outside this interval. Now, introduce the Mobius map:

$$z_3 = f_3(z_2) = \frac{z_2 - 1}{z_2 + 1} \quad (3)$$

This is easily seen to map the upper-half plane to the upper-half plane, with  $z_2 = 1, -1, \infty$  corresponding to  $z_3 = 0, \infty, 1$ , respectively. Thus, now on the real  $z_3$  axis,  $\phi = 1$  for  $z_3 < 0$  and  $\phi = 0$  for  $z_3 > 0$ . Now, introduce

$$\zeta = f_4(\zeta_3) = \frac{1}{\pi} \log z_3 \quad \text{with } \arg z_3 \text{ in } [0, \pi] \quad (4)$$

It is clear that  $f_4$  maps the upper-half plane into a strip of width 1 in the  $\zeta = \xi + i\eta$  plane, where  $\phi = 1$  on  $\eta = 0$  and  $\phi = 0$  on  $\eta = 1$ . A solution to Laplace's equation with the given boundary condition in the  $\zeta$  plane is:

$$\Phi(\xi, \eta) = \eta = \text{Re } i(\xi + i\eta) \quad (5)$$

This means that, without loss of generality,

$$\chi(\zeta) = i\zeta \quad (6)$$

Therefore

$$\Omega(z) = \chi(f_4(f_3(f_2(f_1(z)))))) = \frac{2i}{\pi} \log \left[ \frac{1 + z^2}{1 - z^2} \right] \quad (7)$$

and  $\phi(x, y) = \text{Re } \Omega(x + iy)$  It can be checked that the boundary conditions are indeed satisfied, with suitable choice of cuts and branches. Also, this is the unique solution  $\phi$  in the class of solutions bounded in the closure of the domain, as one can prove using Maximum principle.

### 3 More explicit examples of conformal mapping

**Example 1:** Let  $D$  be the domain in the intersection of two circles, one centered at  $z = 1$  and the other at  $z = i$ , each of radius 2 (See Fig. 3). Determine solution  $\phi$  satisfying the

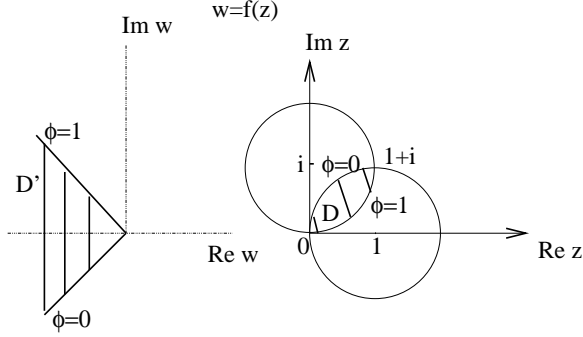


Figure 1: Domain  $D$  in the  $z$  and  $w$ -planes

Laplace's equation in  $D$ , with boundary conditions  $\phi = 0$  on  $\{z : |z - 1| = 2\} \cap \partial D$  and  $\phi = 1$  on the other circular arc.

**Solution:** Because  $D$  is bounded by circular arcs, with intersection points  $z = 0$  and  $z = 1 + i$ , we choose Mobius map

$$w = f(z) = \frac{z}{z - 1 - i} \quad (8)$$

The resulting domain  $D' = f(D)$  is bounded by  $L_1 := \arg w = 3\pi/4$  and  $L_2 := \arg w = 5\pi/4$ , with  $\phi = 1$  on  $L_1$  and  $\phi = 0$  on  $L_2$ . Now, the transformation

$$\zeta = g(W) := \frac{2}{\pi} \left( \log w - i \frac{3\pi}{4} \right) \quad \text{where } \arg w \text{ is in } (0, 2\pi) \quad (9)$$

maps  $D'$  into the strip between  $\eta = 0$  and  $\eta = 1$ , where  $\zeta = \xi + i\eta$ . (See Fig. 3)

Clearly

$$\Phi = (1 - \eta) \quad (10)$$

is a harmonic function in  $(\xi, \eta)$ , implying  $\Phi = \Re \chi(\xi + i\eta)$ , where

$$\chi = 1 + i\zeta, \quad (11)$$

Therefore,  $\phi(x, y) = \Re \Omega(x + iy)$ , where

$$\Omega(z) = 1 + \frac{2i}{\pi} \left( \log \frac{z}{z - 1 - i} - i \frac{3\pi}{4} \right) \quad (12)$$

with appropriate choice of branch of log. On taking the real part, we obtain explicit expression

$$\phi(x, y) = \frac{2}{\pi} \left( \frac{\pi}{4} - \arctan \frac{x - y}{x(x - 1) + y(y - 1)} \right)$$

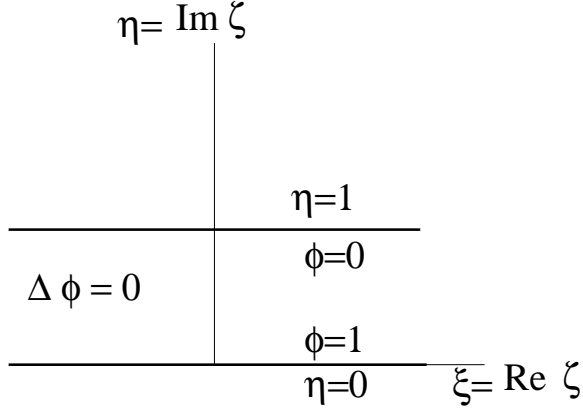


Figure 2: Boundary value problem in the  $\zeta$ -plane

where  $-\pi/2 < \arctan(\cdot) < \pi/2$ .

## 4 More on conformal mapping

Now we address the uniqueness part in the Riemann mapping theorem.

**Lemma 5** *Assume  $D$  is a simply connected domain and  $f : D \rightarrow D_1$  is an analytic homeomorphism. If we require  $f(z_0) = 0$  and  $f'(z_0) > 0$ , then  $f$  is unique.*

PROOF. Assume  $f : D \rightarrow D_1$  is an analytic homeomorphism. Then, consider  $g$  to be another such analytic homeomorphism. Then,  $f \circ g^{-1} : D_1 \rightarrow D_1$  and  $f(g^{-1}(0)) = 0$ , implying that  $f(g^{-1}(\zeta)) = e^{i\alpha}\zeta$ . This implies  $g^{-1}(\zeta) = f^{-1}(e^{i\alpha}\zeta)$  or  $f(z) = e^{i\alpha}g(z)$ . The condition  $f'(0) > 1, g'(0) > 0$ , fixes  $\alpha = 2n\pi$ , or  $f = g$ .  $\square$

**Remark 1** *Notice that instead of requiring  $f(z_0) = 0$  and  $f'(0) > 0$ , the mapping is also unique if we require three specific points on  $\partial D$  to correspond to three points on  $\partial D_1$  in the right order, in terms of orientation. Proof of this statement is similar to the above Lemma, except that  $f \circ g^{-1}$  is a priori more general Mobius map. The parameters in the map are uniquely determined when we require three points of one domain boundary to map to three points on the boundary of the other.*

## 4.1 Schwartz-Christoffel map: An intuitive construction

We now turn to constructing analytic homeomorphism between  $\mathbb{H}^+$  (the upper-half plane) and the interior of a polygon  $D$ . Suppose  $z = f(\zeta)$  is such a map. Let the  $n$  vertices of the polygon, enumerated in some counter clockwise order, be the points  $z_1, z_2, \dots, z_n$  and let their image points on the  $\xi = \Re\zeta$  axis be denoted by  $\xi_1, \xi_2$  through  $\xi_n$ . Without any loss of generality, we take  $\xi_1 < \xi_2 < \xi_3 < \dots < \xi_n$ . Note that this constrains the order the relative orientation of  $z_1, z_2, \dots, z_n$ .

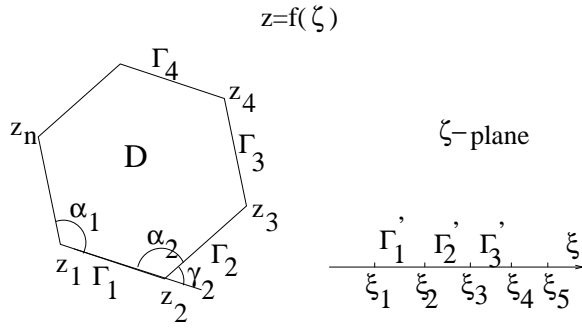


Figure 3: Domain  $D$  in the  $z$ -plane and the upper-half  $\zeta$  plane

The  $n$  sides of the polygon are denoted by  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  (See Fig.4.1), and their respective image in the  $\xi$ -axis given by  $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_n$ . If  $\xi_1 \neq -\infty$  and  $\xi_n \neq +\infty$ ,  $\Gamma'_n$  consists of two segments  $\xi < \xi_1$  and  $\xi > \xi_n$ ). Let the interior angle of the polygon be denoted by  $\alpha_j$ ,  $j = 1, \dots, n$ . Define  $\gamma_j = \pi - \alpha_j$ . Notice that a counter-clockwise rotation of  $\Gamma_{j-1}$  through an angle  $\gamma_j$  aligns it with  $\Gamma_j$ . From properties of polygons,

$$\sum_{j=1}^n \alpha_j = (n-2)\pi \quad \text{or} \quad \sum_{j=1}^n \left(1 - \frac{\alpha_j}{\pi}\right) = \sum_{j=1}^n \frac{\gamma_j}{\pi} = 2 \quad (13)$$

Let  $f : \mathbb{H}^+ \rightarrow D$  be such a map. It is to be noted that  $\arg f'(\zeta)$  is a piecewise constant on each segment  $\Gamma'_j$ . Further, it is to be noted that

$$\arg f' + \sum_{j=1}^n \frac{\gamma_j}{\pi} \arg(\zeta - \xi_j) = \text{constant} \quad (14)$$

is the same constant on each segment  $\Gamma'_j$ , noting that  $\arg(\zeta - \xi_j) \in [0, \pi]$ . It is clear that within each  $\Gamma'_j$ , there is no variation of the function on the left of (14). As  $\zeta$  moves

from  $\Gamma'_{k-1}$  to  $\Gamma'_k$ ,  $\arg f'$  clearly increases by  $\gamma_k$ , while the summation decreases by  $-\gamma_k$ , since the term  $\arg(\zeta - \zeta_k)$  in the summation changes from  $\pi$  to 0, while other terms in the summation undergoes no change. This suggests that for some constant  $\tilde{a}$ ,

$$f'(\zeta) = \tilde{a} \prod_{j=1}^n (\zeta - \zeta_j)^{-\gamma_j/\pi}$$

implying

$$f(\zeta) = z_1 + \tilde{a} \int_{\xi_1}^{\zeta} d\zeta' \prod_{j=1}^n (\zeta' - \zeta_j)^{-\gamma_j/\pi} \quad (15)$$

We now directly verify that  $f : \partial\mathbb{H}^+ \rightarrow \partial D$  in a 1-1 manner provided some some geometric constraints are satisfied.

Clearly,  $f(\xi_1) = z_1$ . Further, if  $\zeta \in \Gamma'_1$ , it is clear from (15) that for some complex  $a$ , related to  $\tilde{a}$ ,

$$z = f(\zeta) = z_1 + a e^{i\gamma_1} \int_{\xi_1}^{\zeta} d\zeta' (\zeta' - \xi_1)^{-\gamma_1/\pi} \prod_{j=2}^n (\xi_j - \zeta')^{-\gamma_j/\pi} \quad (16)$$

Using (15), it is clear that on this segment  $\Gamma'_1$ ,  $\arg(z - z_1)$ , as computed from the above formula comes out to be constant,  $\arg a + \gamma_1$ . Evaluating (16) at  $\zeta = \xi_2$ , one obtains

$$z_2 - z_1 = a e^{i\gamma_1} \int_{\xi_1}^{\xi_2} d\zeta (\zeta - \xi_1)^{-\gamma_1/\pi} \prod_{j=2}^n (\xi_j - \zeta)^{-\gamma_j/\pi} \quad (17)$$

On the other hand, if  $\zeta \in \Gamma'_2$ ,

$$z = f(\zeta) = z_2 + a e^{i\gamma_1 + i\gamma_2} \int_{\xi_2}^{\zeta} d\zeta' (\zeta' - \xi_1)^{-\gamma_1/\pi} (\zeta' - \xi_2)^{-\gamma_2/\pi} \prod_{j=3}^n (\xi_j - \zeta')^{-\gamma_j/\pi} \quad (18)$$

So, in this segment  $\arg(z - z_2) = \arg a + \gamma_1 + \gamma_2$  in accordance to what it should be (See Fig.??) Evaluating (??) at  $\zeta = \xi_3$ , we obtain

$$z_3 - z_2 = a e^{i\gamma_1 + i\gamma_2} \int_{\xi_2}^{\xi_3} d\zeta (\zeta - \xi_1)^{-\gamma_1/\pi} (\zeta - \xi_2)^{-\gamma_2/\pi} \prod_{j=3}^n (\xi_j - \zeta)^{-\gamma_j/\pi} \quad (19)$$

Generally, we can verify that on  $\Gamma'_k$ ,

$$\arg(z - z_k) = \arg a + \sum_{j=1}^k \gamma_j$$

Now, if the lengths of the sides of the polygons are  $l_1, l_2, l_n$ , we must have from (17), (19) and similar relations for  $z_{k+1} - z_k$ , the following constraints:

$$l_k = |a| \int_{\xi_k}^{\xi_{k+1}} d\zeta \prod_{j=1}^k (\zeta - \xi_j)^{-\gamma_j/\pi} \prod_{j=k+1}^n (\xi_j - \zeta)^{-\gamma_j/\pi} \quad (20)$$

Equations (20) constitute  $n - 1$  constraints involving  $n + 1$  parameters  $\xi_1, \xi_2, \dots, \xi_n$  and  $|a|$ , once a specified  $\zeta = \xi_1$  corresponds to  $z = z_1$ . The additional two real degrees of freedom may be used by that the specifying values of  $\xi_2, \xi_3$  for instance, or perhaps specifying  $|a|, \xi_2$ . The three degrees of freedom in this special case is a reflection of the three degrees of freedom one has from the general Riemann mapping theorem.

Seemingly, there is an additional relation

$$l_n = |a| \left( \int_{\xi_n}^{+\infty} d\zeta \prod_{j=1}^n (\zeta - \xi_j)^{-\gamma_j/\pi} + \int_{-\infty}^{\xi_1} d\zeta \prod_{j=1}^n (\xi_j - \zeta)^{-\gamma_j/\pi} \right) \quad (21)$$

However, (21) is not independent of relations (20) and indeed is implied by them. To see this, we note that

$$\int_{-\infty}^{\infty} d\zeta f'(\zeta) = 0$$

for a contour that avoids the singular points  $\xi_j$  by going over them. To see this, we note that  $f' \sim \text{constant } \zeta^{-2}$  as  $\zeta \rightarrow \infty$ . Therefore,

$$z_1 - z_n = \left( \int_{\xi_1}^{\xi_n} + \int_{-\infty}^{\xi_1} \right) f'(\zeta) d\zeta = - \sum_{j=1}^{n-1} \int_{\xi_j}^{\xi_{j+1}} f'(\zeta) d\zeta$$

Therefore, (21), which is the magnitude of  $z_1 - z_n$  is implied by the set of relations (20).

**Remark 2** *The Schwarz-Christoffel formula (15) simplifies by setting one  $\xi_n = \infty$ . In a formal procedure, it is seen that for fixed  $\zeta$ , as  $\xi_n \rightarrow \infty$  in a manner so that  $a (-\xi_n)^{-\gamma_n/\pi} \rightarrow \tilde{a}$ , then*

$$z = z_1 + \tilde{a} \int_{\xi_1}^{\zeta} d\zeta' \prod_{j=1}^{n-1} (\zeta - \xi_j)^{-\gamma_j/\pi} \quad (22)$$

*It can be directly verified that (22) maps the real axis into the polygon under consideration, provided the subsidiary length constraints in (??) are satisfied. Equation (22) is simpler usually than the original expression as the factor  $(\zeta - \xi_n)^{-\gamma_n/\pi}$  is entirely missing.*

**Remark 3** That the analytic homomorphism  $f : \mathbb{H}^+ \rightarrow D$  must be in the form (15) may be proved by repeated use of Schwartz reflection principle (see Nehari or O.Costin Notes). We sketch the outline of the proof briefly. Define the expression in (15) as  $\tilde{f}$ . Assume the segment  $\arg(z - z_1) = \phi_1$  on line segment  $\Gamma_1$ . Then, we can show that near  $\zeta = \xi_1$

$$g_1(\zeta) \equiv \left\{ e^{-i\phi_1} [f(\zeta) - z_1] \right\}^{\pi/\alpha_1}$$

is analytic in the upper-half  $\zeta$  plane, continuous upto the boundary with vanishing imaginary part on the real axis segment that contains  $\xi_1$  in the interior. Schwarz-reflection principle guarantees no singularity of  $g_1(\zeta)$  at  $\zeta = \xi_1$  as is true for

$$\tilde{g}_1(\zeta) \equiv \left\{ e^{-i\phi_1} [\tilde{f}(\zeta) - z_1] \right\}^{\pi/\alpha_1}$$

and analytically continuable to the lower-half plane. We can repeat similar argument at each  $\xi_j$  and eventually arrive at the conclusion that  $f'(\zeta)/\tilde{f}'(\zeta)$  is singularity free in the entire complex plane and is bounded at  $\infty$ , thereby leading to  $f'(\zeta) = C\tilde{f}'$ . With suitable normalization intrinsic in the the relation of  $a$  to length of polygonal sides,  $C = 1$ .

## 4.2 More applications of conformal map

**Example 1:** Find an analytic homeomorphism that maps the interior of the rectangle with vertices at the points  $z = \pm 1, z = \pm 1 + i c$  for  $c > 0$  into the upper-half  $\zeta$  plane (See Fig. 4.2)

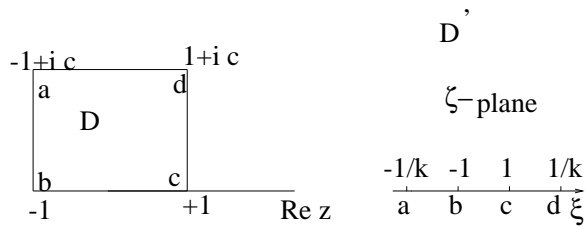


Figure 4: Domain  $D$  in the  $z$  and  $\zeta$  plane

**Solution:** We choose the three degrees of freedom by requiring that  $\zeta = \pm 1$  to correspond to  $z = \pm 1$ , and  $\zeta = \pm \frac{1}{k}$  (with  $0 < k < 1$  to correspond to  $z = \pm 1 + i c$ , respectively. Note that we have not specified  $k$ , as that would have corresponded to specifying four image points in the  $\zeta$ -domain. Then, it is clear from Schwarz Christoeffel-formula that

$$z = -1 + b \int_{-1}^{\zeta} \frac{d\zeta}{\sqrt{1-\zeta^2} \sqrt{1-k^2\zeta^2}} \quad (23)$$

where  $b > 0$ , and  $\arg(1 - \zeta)$ ,  $\arg(1 - k\zeta)$  are each in  $[-\pi, 0)$ , while  $\arg(\zeta + 1)$ ,  $\arg(1 + k\zeta) \in [0, \pi]$ . The length constraints become:

$$2 = b \int_{-1}^1 \frac{d\zeta}{\sqrt{1-\zeta^2} \sqrt{1-k^2\zeta^2}}$$

$$c = b \int_1^{1/k} \frac{d\zeta}{\sqrt{\zeta^2-1} \sqrt{1-k^2\zeta^2}}$$

**Remark:** If we introduce the elliptic integral of the first kind:

$$E(k, \zeta) = \int_0^\zeta \frac{d\zeta}{\sqrt{1-\zeta^2} \sqrt{1-k^2\zeta^2}} \quad (24)$$

It is clear that in the previous example

$$z = b E(k, \zeta) \quad (25)$$

The inverse of this is known in the elliptic function theory to be the Jacobian elliptic function  $sn$  (Whittaker & Watson, "Modern Analysis", Chap. 22). In our case,

$$\zeta = sn\left(\frac{z}{b}, k\right) \quad (26)$$

This is not surprising at all since on differentiation of expression of  $z = E(k, \zeta)$ , it follows that  $\zeta(z)$  satisfies the first order nonlinear ODE

$$\left(\frac{d\zeta}{dz}\right)^2 = (1-\zeta^2)(1-k^2\zeta^2)$$

whose solution is known to be the Elliptic function.

**Remark 4** In the asymptotic limit  $k \rightarrow 0$ , we obtain the semi-infinite strip obtained by letting  $c \rightarrow \infty$ . In that case, the mapping function simplifies to

$$z = -1 + b \int_{-1}^\zeta (1-\zeta^2)^{-1/2} d\zeta = -1 + b \sin^{-1} \zeta + b \frac{\pi}{2} \quad (27)$$

with choice of branch of  $\sin^{-1}$  so that  $\sin^{-1} 0 = 0$  the constraint on length translates to  $2 = b \pi$ . So,

$$z = f(\zeta) = \frac{2}{\pi} \sin^{-1} \zeta$$

It's inverse is

$$\zeta = h(z) = \sin \frac{\pi z}{2}$$

Recall that sometimes back, we showed directly the property that  $\sin$  maps a semi-infinite strip into the upper-half plane.

**Example 2:** Let  $D$  be the domain outside the unit circle (See Fig. 4.2). Find a solution  $\phi(x, y)$  that satisfies Laplace's equation in  $D'$ , except at point  $(x_0, y_0)$ , which is a point vortex of strength  $p$ , and as  $x \rightarrow \pm \infty$ ,

$$\frac{\phi(x, y)}{x} \rightarrow U \tag{28}$$

and on the unit circular boundary,

$$\frac{\partial \phi}{\partial n} = 0 \tag{29}$$

**Definition 6** If  $\phi(x, y) = \Re \Omega(x + iy)$  for some analytic  $\Omega$ , then a point vortex at  $z = z_0$  of strength  $p$  is a singularity of  $\Omega$  such that  $\Omega(z) - ip \log(z - z_0)$  is regular and single valued at  $z = z_0$ .

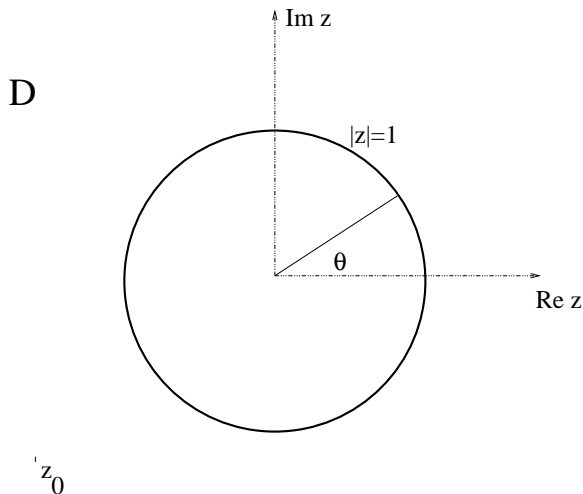


Figure 5: Potential problem outside a circle

**Solution:** Let  $\Omega(z)$  be the corresponding complex potential, with

$$\phi(x, y) = \text{Re} [\Omega (x + iy) ] = \text{Re} [ \phi + i\psi ]$$

From the C-R conditions, (29) is equivalent to

$$\frac{\partial \psi}{\partial \theta} = 0 \quad ; \quad \text{on integration } \psi = c \quad (30)$$

on  $z = e^{i\theta}$ , for some constant  $c$ , taken to be zero w.l.o.g. A point vortex at  $z_0 = x_0 + iy_0$  implies

$$\Omega - ip \log (z - z_0) \quad (31)$$

is regular and single valued as  $z \rightarrow z_0$ . Since  $\psi = 0$  on the boundary of  $|z| = 1$ , Schwarz reflection principle holds, if  $\Omega(z)$  is continuous upto  $|z| = 1$ , as will be assumed. Thus, points  $z = 0$  and  $z = \frac{1}{z_0^*}$ , which are image of  $z = \infty$  and  $z = z_0$  across  $|z| = 1$  (i.e. via inversion), are possible singular points of  $\Omega(z)$  inside  $|z| = 1$ . Thus, consider

$$g(z) = \Omega(z) - U \left[ z + \frac{1}{z} \right] - ip \log \frac{z - z_0}{1 - zz_0^*} - i\Gamma \log z \quad (32)$$

It is easily verified that  $\Im g = 0$  on  $|z| = 1$  and that it has removable singularity at  $z = z_0$  and  $z = 1/z_0^*$ .  $g(z)$  is an entire function. Note  $|g(z)|/|z|$  must be bounded from condition (28) and so  $g(z)$  is at best be a linear function of  $z$ . But (29) determines the coefficient of  $z$  as  $z \rightarrow \infty$ . Therefore, except for an additive constant,

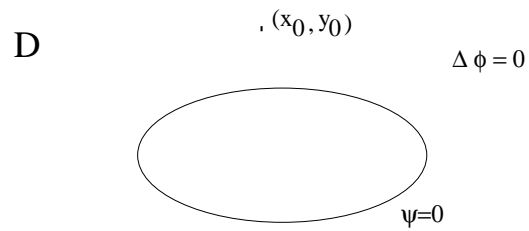
$$\Omega(z) = U \left[ z + \frac{1}{z} \right] + ip \log \frac{z - z_0}{1 - zz_0^*} + i\Gamma \log z \quad (33)$$

The real part of this gives  $\phi$ . Note that  $\Gamma$  is an arbitrary constant, and hence there is non-uniqueness for two dimensional Laplace's equation unless the total circulation  $\Gamma + p$  at  $\infty$  is specified.

**Remark 5** *The above can be used to find determine the solution to Laplace's equation outside an ellipse for instance (see Fig.4.2), where a point vortex is located at  $z_0 = (x_0, y_0)$  within the domain. Recall a suitable Joukowski's transformation maps the outside of an ellipse to the outside of a circle. Further, a point vortex remains a point vortex after a conformal transformation since if  $[\Omega - ip \log (z - z_0)]$  has a removable singularity at  $z = z_0$ ,  $\chi(\zeta) - ip \log (\zeta - \zeta_0)$  also has a removable singularity at  $\zeta = \zeta_0$ , where  $z = f(\zeta)$  and  $\chi(\zeta) = \Omega(f(\zeta))$ . This follows by noting that from Taylor expansion about  $\zeta_0$ ,*

$$z - z_0 = f(\zeta) - f(\zeta_0) = (\zeta - \zeta_0) h(\zeta)$$

where  $h(\zeta)$  is analytic and nonzero at  $\zeta = \zeta_0$ .



$$\phi \sim U x$$

Figure 6: Harmonic  $\phi$  outside an ellipse