Week 8 lectures

1. Riemann Hilbert Problem:

Riemann-Hilbert problem refers to determination of a sectionally Holomorphic function \((i.e.\) analytic everywhere off a curve) \(\Phi(z)\) so that across a simple smooth curve \(C\), \(\Phi\) jumps by a specified amount \(\phi\), \(i.e.\)

\[
\Phi_+(t) - \Phi_-(t) = \phi(t) \quad \text{for } t \in C
\]

where subscript \(+\) and \(-\) indicates limiting values of \(\Phi\) as point \(t\) on the curve is approached from different sides.

Figure 1. \(z\) approaching a point \(t\) on a simple smooth oriented curve \(C\) from the left \((+\) side.

Remark 1.1. Riemann-Hilbert problem arises in a wide variety of problems: such as solving integral equations of certain types, inverse scattering problems, nonlinear integrable systems, radon transforms that arise in tomography, just to name a few.

Remark 1.2. A particular case of Riemann-Hilbert problem is when \(C\) is the entire real line. In that case (or in any other case of closed contour \(C\)) \(\Phi_+\) and \(\Phi_-\) are two different analytic functions in the upper-half and lower-half plane (two sides of \(C\)) with specified jump:

\[
\Phi_+(x) - \Phi_-(x) = f(x)
\]
We may write this relation in terms of distribution
\begin{equation}
\frac{\partial \Phi}{\partial y} = f(x)\delta(y)
\end{equation}

A generalization of this is the \(\bar{\delta}\) (\(D\)-bar) problem, when we seek \(\Phi\) satisfying
\begin{equation}
\frac{\partial \Phi}{\partial \bar{z}} = g(x, y)
\end{equation}
in some region \(D \subset \mathbb{C}\), where
\[\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left\{ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right\}\]

**Definition 1.3.** A function \(\phi\) is said to be Holder-Continuous on a smooth curve \(C\) of order \(\lambda \in (0, 1)\), if for any \(x, y \in C\),
\[\phi(x) - \phi(y) = O \left( |x - y|^\lambda \right)\]

Let \(C = \{z : z = \gamma(t) \ 0 \leq t \leq 1\}\) be a simple smooth oriented curve. For a closed curve, we will always take the orientation to be counterclockwise. At each point \(\gamma(t)\), for \(t \in (0, 1)\), we draw a small enough circle so that the circle intersects \(C\) at just two points. The curve bisects the circle into two parts, each of which is a near semi-circle. If we approach \(C\) from the interior of a near semi-circle that lies to the left of \(C\), then we say that \(C\) has been approached from the left. Similarly, we define approach from the right. Limiting values of a function from the left and right of an oriented simple curve \(C\) will be always denoted by subscript + and − respectively. Note \(C\) may either be open or closed. In the latter case, \(\gamma(0) = \gamma(1)\).

We consider the following function(s) \(\Phi(z)\) for \(z \notin C\):
\begin{equation}
\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s - z} ds
\end{equation}

Now, note from Fubini’s theorem, \(\oint \Phi(z)dz = 0\) for any close contour in the \(z\) plane as long as this contour is at a finite distance from \(C\). So, from Moerara’s theorem \(\Phi\) is analytic off \(C\), i.e. \(\Phi\) is a sectionally holomorphic function. Note in the case of closed contour \(C\), the above formula defines two distinct function \(\Phi_+(z)\) in the domain on the left of \(C\) and \(\Phi_-(z)\) in the domain to the right of \(C\).

We also notice from (1.5) that for any bounded curve \(C\), the asymptotic behavior
\begin{equation}
\Phi(z) \sim -\frac{1}{2\pi iz} \int_C \phi(s)ds \quad \text{as} \quad z \to \infty
\end{equation}
Theorem 1.1. Plemelj formulae Assume $\phi$ is a holder continuous function on a simple smooth compact curve $C$, which may or may not be closed. Let $t \in C \setminus \gamma(0) \cup \gamma(1)$ (i.e. not end points). Let $z_n$ be a set of points approaching $t$ from the left (or right). Then, let $\Phi_\pm(t) = \lim_{n \to \infty} \Phi(z_n)$, with $\Phi$ defined by (1.3). We have

$$\Phi_\pm(t) = \pm \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \oint_{C} \frac{\phi(s)}{s - t} ds.$$ 

Remark 1.4. We consider the special case where $C$ coincides with part of the real axis, $[-1, 1]$ (see Fig. 2), without much loss of generality. More general smooth curves may be handled by introducing a smooth change in variable that parametrizes the curves. We will prove Plemelj formulae at the end of some preliminary lemmas.

![Figure 2](image_url)

**Figure 2.** $z_n = t_n + i y_n$ approaching $t \in (-1, 1)$ from the left (top), with $t_n \to t$, $y_n \to 0^+$

Lemma 1.5. Define $\Psi_n(s) = \frac{\phi(s) - \phi(t_n)}{s - z_n}$, where $z_n = t_n + iy_n \equiv t + \epsilon_n + iy_n$, where $\epsilon_n \to 0$ and $y_n \to 0^+$ (or $y_n \to 0^-$, when approach is from the right) as $n \to \infty$. If $\phi$ is Holder continuous, then

$$\lim_{n \to \infty} \int_{-1}^{1} \Psi_n(s) ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds.$$ 

Proof. We note that

$$\int_{-1}^{1} \Psi_n(s) ds = \int_{-2}^{2} \Psi_n(s) \chi(s) ds,$$

where $\chi$ is the characteristic function for the interval $[-1, 1]$, i.e. $\chi(s) = 1$ in this interval and 0 outside. Then,

$$\int_{-1}^{1} \Psi_n(s) ds = \int_{-2}^{2} \frac{\phi(s) - \phi(t + \epsilon_n)}{s - t - \epsilon_n - iy_n} \chi(s) ds = \int_{-2}^{2} \frac{\phi(\sigma + \epsilon_n) - \phi(t + \epsilon_n)}{\sigma - t - iy_n} \chi(\sigma + \epsilon_n) d\sigma$$
However, from Holder Condition,

$$|\phi(\sigma + \epsilon_n) - \phi(t + \epsilon_n)| \leq C|\sigma - t|^{\lambda},$$

and since $\frac{1}{|\sigma - t - i\epsilon_n|} \leq \frac{1}{|\sigma - t|}$ we have

$$|\int_{-1}^{1} \Psi_n(s) ds| \leq C \int_{-2}^{2} |\sigma - t|^{\lambda-1} d\sigma < \infty,$$

since $\lambda > 0$. From dominated convergence theorem, it follows that

$$\lim_{n \to \infty} \int_{-1}^{1} \Psi_n(s) ds = \int_{-1}^{1} \frac{\phi(\sigma) - \phi(t)}{\sigma - t} \chi(\sigma) d\sigma = \int_{-1}^{1} \frac{\phi(\sigma) - \phi(t)}{\sigma - t} d\sigma$$

Lemma 1.6. For $t \in (-1, 1)$ with $y_n \to 0^\pm$, $\epsilon_n \to 0$ and $t_n = t + \epsilon_n$,

$$\lim_{n \to \infty} \int_{-1}^{1} \frac{1}{s - t_n - iy_n} ds = \int_{-1}^{1} \frac{1}{s - t} ds \pm \pi i$$

![Figure 3. Deformed contour for evaluation of $\lim_{n \to \infty} \int_{-1}^{1} \frac{1}{s - t_n - iy_n} ds$, when $y_n \to 0^+$](image)

Proof. Since the integrand is analytic in $s$ when $y_n > 0$, we deform the contour to the one shown in Fig. 3, where $\epsilon > 0$ is chosen small enough so that $-1 < t - \epsilon < t + \epsilon < 1$. We get

$$\int_{-1}^{1} \frac{1}{s - t_n - iy_n} ds = \left\{ \int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1} \right\} \frac{ds}{s - t_n - iy_n} + \int_{C_\epsilon} \frac{ds}{s - t_n - iy_n}$$

As $n \to \infty$, since $\epsilon_n \to 0$, $t_n \to t$, we have due to uniform convergence,

$$\lim_{n \to \infty} \left\{ \int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1} \right\} \frac{ds}{s - t_n - iy_n} = \left\{ \int_{-1}^{t-\epsilon} + \int_{t+\epsilon}^{1} \right\} \frac{ds}{s - t}$$

Also, for $y_n \to 0^+, t_n \to t$, from uniform convergence of integrand,

$$\lim_{n \to \infty} \int_{C_\epsilon} \frac{ds}{s - t_n - iy_n} = \int_{C_\epsilon} \frac{ds}{s - t} = \int_{-\pi}^{\pi} \frac{i e^{i\theta}}{e^{i\theta}} d\theta = \pi i$$
On the other hand if \( y_n \to 0^- \), we choose \( C_\epsilon \) to be small upper-semicircular contour to get,

\[
\lim_{n \to \infty} \int_{C_\epsilon} \frac{ds}{s - t_n - iy_n} = \int_{C_\epsilon} \frac{ds}{s - t} = \int_0^1 i\epsilon e^{i\theta} d\theta = -\pi i
\]

Therefore, for \( y \to 0^\pm \),

\[
\lim_{n \to \infty} \left\{ \int_{t-\epsilon}^{t+\epsilon} + \int_{t+\epsilon}^{t+1} \right\} \frac{ds}{s - t_n - iy_n} = \left\{ \int_{t-\epsilon}^1 + \int_{t+\epsilon}^{t+1} \right\} \frac{1}{s - t} ds \pm \pi i
\]

Taking the limit of \( \epsilon \to 0 \), we get the desired result.

**Proof of Theorem 1.1:**

We note that

\[
\int_{-1}^1 \phi(s) ds = \int_{-1}^1 \phi(s) - \phi(t) ds + \phi(t) \int_{-1}^1 \frac{ds}{s - z_n} ds
\]

\[
= \Psi_n + \phi(t) \int_{-1}^1 \frac{ds}{s - t_n - iy_n}
\]

Using Lemmas (1.5) and (1.6), we obtain the desired proof.

**Remark 1.7.** With additional conditions on decay of \( \phi \), Plemelj formula can be extended to \( C \) which is not compact as in the following Lemma.

**Lemma 1.8.** Extension of Plemelj when \( C = \mathbb{R} \)

Assume \( C = \mathbb{R} \) (the real line). Assume that as \( |s| \to \infty \), \( \phi(s) \to L \) and that \( \phi \) is Holder continuous uniformly for any \( x, y \in \mathbb{R} \) and that

\[
|\phi(s) - L| = O(s^{-\mu}), \text{ as } |s| \to \infty \text{ for } \mu > 0
\]

Then Theorem 1.1 holds.

**Proof.** is similar to that of 1.1 and is left as an exercise.

**Remark 1.9.** We note that Plemelj Formula immediately implies the jump formula:

\[
\Phi_+(t) - \Phi_-(t) = \phi(t)
\]

**Remark 1.10.** In the case, \( C \) a closed curve (i.e. separates the plane into two separate + and - regions),

\[
\frac{1}{2\pi i} \oint_C \frac{\phi(s) ds}{s - z}
\]

actually defines two distinct analytic functions \( \Phi_+(z) \) and \( \Phi_-(z) \) depending on whether \( z \) is on the plus (left) or minus side (right) of the oriented curve \( C \). These functions must be distinct because the Plemelj...
formulae (9) demands that they are unequal, when each is continued to $C$ itself.

**Remark 1.11.** Note that the analytic continuation of $\Phi(z)$ (or $\Phi_{\pm}(z)$ for closed curve) is possible across the curve $C$ when $\phi(z)$ is analytic. This analytic continuation is given by

$$\frac{1}{2\pi i} \oint_{C} \frac{\phi(s)}{s - z} \, ds \pm \phi(z)$$

depending on whether the analytic continuation proceeded from the left of the curve $C$ to its right, or the other way around.

**Simple example:** Let $C$ be a unit circle. Determine $\Phi_{+}(z)$ and $\Phi_{-}(z)$ so that for $t \in C$, $\Phi_{+}(t) - \Phi_{-}(t) = 1$. We write

$$\Phi_{\pm}(z) = \frac{1}{2\pi i} \oint_{|s|=1} \frac{ds}{s - z}$$

for $z$ inside and outside the unit circle, respectively. For $|z| < 1$, clearly, $\Phi_{+}(z) = 1$; while for $|z| > 1$, $\Phi_{-}(z) = 0$. Let $C$ be a unit circle. Determine $\Phi_{+}(z)$ and $\Phi_{-}(z)$ so that for $t \in C$, $\Phi_{+}(t) - \Phi_{-}(t) = 1$. We write

$$\Phi_{\pm}(z) = \frac{1}{2\pi i} \oint_{|s|=1} \frac{ds}{s - z}$$

for $z$ inside and outside the unit circle. For $|z| < 1$, clearly, $\Phi_{+}(z) = 1$; while for $|z| > 1$, $\Phi_{-}(z) = 0$.

**Another example:** Consider the problem of finding a sectionally holomorphic function $\Phi(z)$ in $\mathbb{C} \setminus [-1, 1]$ satisfying

$$\Phi_{+}(x) - \Phi_{-}(x) = 1 \text{ for } x \in (-1, 1)$$

We know that

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{dt}{t - z} = \frac{1}{2\pi i} \ln \left\{ \frac{z - 1}{z + 1} \right\}$$

will satisfy this condition. We can check directly. Approaching $x \in (-1, 1)$ from the top, $\ln(z - 1) \to \ln(1 - x) + i\pi$, while $\ln(z + 1) \to \ln(1 + x)$. Therefore, from the top,

$$\frac{1}{2\pi i} \ln \left\{ \frac{z - 1}{z + 1} \right\} \to \frac{1}{2\pi i} \ln \left\{ \frac{1 - x}{1 + x} \right\} + \frac{1}{2}$$

From the bottom, since $\ln(z - 1) \to \ln(1 - x) - i\pi$, it follows

$$\frac{1}{2\pi i} \ln \left\{ \frac{z - 1}{z + 1} \right\} \to \frac{1}{2\pi i} \ln \left\{ \frac{1 - x}{1 + x} \right\} - \frac{1}{2}$$

Therefore, indeed $\Phi_{+}(x) - \Phi_{-}(x) = 1$. 
Remark 1.12. The answer in the last example is not unique. We can add to a particular $\Phi(z)$ any analytic function of $z$ with isolated singularities at end points. If we required $\lim_{z \to z_e} (z - z_e) \Phi(z) = 0$ at each end point $z_e$ and in addition required $\lim_{z \to \infty} \Phi(z) = 0$, then the answer would be unique. To see this, let $\Psi(z)$ be another solution. Then $\Phi(z) - \Psi(z)$ is single valued at $z = z_e$ since each of $\Psi$ and $\Phi$ jumps by the same amount $\psi$. Since the condition at $z_e$ rules out simple pole, it follows $\Phi - \Psi$ is an entire function. Growth condition at $\infty$ implies $\Phi - \Psi = 0$.

2. Scalar Homogeneous Riemann-Hilbert (RH) Problem

In this case we seek to find sectionally holomorphic function $\Phi(z)$ so that on a simple smooth curve $C$:

$$
\Phi_+(t) = g(t)\Phi_-(t)
$$

where $g(t) \neq 0$ on $C$ and Holder continuous. By taking the log of both sides, this becomes a problem of determining $\Psi(z)$ so that $\Psi_+(t) - \Psi_-(t) = \ln g(t)$. This becomes a familiar problem as discussed in the last section. Once $\Psi(z)$ is determined, we have $\Phi(z) = e^{\Psi(z)}$. When $C$ is closed there is additional complication since $\ln g(t)$ may not return to the same value as we traverse the closed path $C$. We have to define some new concepts.

Definition 2.1. Let $C : \{z : z = \gamma(t), \ t \in [0, 1]\}$ be a simple smooth closed oriented curve and $\phi$ be a differentiable function on $C$ and with $\phi \neq 0$ on $C$. Then the index of $\phi$ with respect to curve $C$ is an integer defined by

$$
\text{ind}_C \phi \equiv \frac{1}{2\pi i} \int_C \frac{\phi'(s)}{\phi(s)} ds
$$

Remark 2.2. If $\phi$ is a meromorphic function, we know index is $N - P$, the difference of number of zeros and number of poles within $C$. However, $\text{ind}_C \phi$ is defined even if the function is not an analytic function anywhere since since $\frac{1}{2\pi i} [\ln \phi]_{\text{jump}}$ must be an integer if we follow the $\ln$ function on branch where it is continuous.

Definition 2.3. A function $\Phi$ has degree $k$ at $\infty$ if there exists some constant $C \neq 0$ so that

$$
\Phi(z) = Cz^k + O(z^{k-1}) \text{ as } z \to \infty
$$

A function $\Phi$ is said to have a finite degree at $\infty$ if there exists some finite $m$ so that $\Phi(z) = o(z^m)$ as $z \to \infty$. 
2.1. Solution to homogeneous RH problem.

**Lemma 2.4.** Let $C$ be a simple smooth closed oriented curve with $\text{ind}_C g = k$ for some nonzero differentiable function $g$ on $C$. Assume $z = 0$ is a point inside $C$. Then a solution to the scalar RH problem (2.7) is given by

$$
\Phi^+ = e^{\Psi_+(z)}, \quad \Phi^- = z^{-k} e^{\Psi_-(z)}
$$

where

$$
\Psi_\pm(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} ds
$$

for $z$ inside and outside $C$ respectively,

where $f(t) = \log [t^{-k} g(t)]$.

**Proof.** Clearly, by taking the log of both sides of (2.7) we can write

$$
\ln \Phi^+ - \ln [t^k \Phi^-] = \ln [t^{-k} g(t)] = f(t)
$$

Since the index of $g$ is $k$, it follows that $f(t)$ is continuous on $C$, since both log $g$ and log $t^k$ jump by $2\pi i k$ as we traverse the curve. Therefore, if we define $\Psi_+(z) = \ln \Phi_+(z)$ for $z$ inside $C$ and $\Psi_-(z) = \ln \{z^k \Phi_-(z)\}$, it follows that

$$
\Psi_+(t) - \Psi_-(t) = f(t)
$$

We know a solution to this is given by

$$
\Psi(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} ds
$$

from which the Lemma follows.

**Remark 2.5.** If $z = 0$ is not inside $C$, we replace the $z^\pm k$, $t^\pm k$ in the above formulae by $(z-z_0)^\pm k$ and $(t-z_0)^\pm k$ for some $z_0$ inside $C$.

**Remark 2.6.** The solution for $\Phi^\pm(z)$ in the last Lemma is not unique. We can clearly multiply each of $\Phi^\pm(z)$ by any entire function $E(z)$ and we still satisfy (2.7). However, if we require that $\Phi_-$ (whose domain includes $\infty$) has degree $m$ at $\infty$, then the most general solution to (2.7) is

$$
\Phi^+ = P_{m+k}(z) e^{\Psi_+(z)}, \quad \Phi^- = P_{m+k}(z) z^{-k} e^{\Psi_-(z)}
$$

where $P_{m+k}(z)$ is a polynomial of degree $m + k$, if $m + k \geq 0$ and zero otherwise. This is because as $z \to \infty$, from (1.6), $\Psi_-(z) = O(1/z)$, and therefore $e^{\Psi^-(z)} \to 1$. Recalling that an entire function that grows no faster than $z^{m+k}$ for $m + k \geq 0$ must be a Polynomial $P_{m+k}$, our claim follows.
3. More on RH problems and open curves

**Lemma 3.1.** If $\phi$ is Holder-continuous on a simple smooth oriented curve $C$, including end points $z_1$ and $z_2$ (See Fig. 2) and

$$
\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s) \, ds}{s - z}
$$

Then, as $z \to z_1$,

$$
\Phi(z) \sim -\frac{1}{2\pi i} \phi(z_1) \ln (z - z_1) + \text{constant}
$$

and as $z \to z_2$,

$$
\Phi(z) \sim \frac{1}{2\pi i} \phi(z_2) \ln (z - z_2) + \text{constant}
$$

**Proof.** Let $t \in C$ be a point on $C$ closest to $z$. Then, we may write

$$
\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(t)}{s - z} \, ds + \frac{\phi(t)}{2\pi i} \int_C \frac{ds}{s - z}
$$

$$
= \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(t)}{s - z} \, ds + \frac{\phi(t)}{2\pi i} \ln \left(\frac{z - z_2}{z - z_1}\right),
$$

with choice of $C$ as the branchcut for ln function. However, if $z$ is sufficiently close to smooth curve $C$, it is clear from geometric consideration.
that $|z - s| \geq |t - s|$. Therefore,

$$\left| \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(t)}{s-z} ds \right| \leq \frac{A}{2\pi} \int_C |s-t|^{\lambda-1}|ds| < \infty$$

From dominating convergence Theorem, it follows that if we take a sequence $z = z_n \rightarrow z_1$, we obtain

$$\lim_{z_n \rightarrow z_1} \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(t)}{s-z} ds = \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(z_1)}{s-z_1} ds = C_1$$

for some constant $C_1$. Hence, combining previous result, as $z = z_n \rightarrow z_1$.

$$\lim_{z_n \rightarrow z_1} \left\{ \frac{1}{2\pi i} \int_C \frac{\phi(s) - \phi(t)}{s-z} ds + \frac{\phi(t)}{2\pi i} \ln \left( \frac{z-z_2}{z-z_1} \right) \right\} \sim -\frac{\phi(z_1)}{2\pi i} \ln(z-z_1) + \frac{\phi(z_1)}{2\pi i} \ln(z_1-z_2) + C_1$$

The second result follows in a similar manner.

**Lemma 3.2.** For a Holder Continuous nonzero $g$ on an open smooth simple oriented $C$, there is unique choice of integers $n_1$ and $n_2$ so that Riemann-Hilbert problem of finding sectionally holomorphic $W(z)$ off $C$ satisfying the following on $C$:

$$W_+(t) = g(t)W_-(t)$$

is of the form

$$W(z) = (z-z_1)^{n_1}(z-z_2)^{n_2} \exp[\Psi(z)],$$

with

$$\Psi(z) = \frac{1}{2\pi i} \int_C \frac{\ln g(s)}{s-z} ds$$

and satisfying

$$W(z) \sim \text{Constant}(z-z_1)^{-\gamma_1} \text{ as } z \rightarrow z_1 \text{ for } \gamma_1 \in (0,1)$$

$$W(z) \sim \text{Constant}(z-z_2)^{-\gamma_2} \text{ as } z \rightarrow z_2 \text{ for } \gamma_2 \in (0,1)$$

**Proof.** We already know from last week that $e^{\Psi(z)}$ is a solution to the scalar homogeneous Riemann-Hilbert problem, recalling that $\ln W_+(t) - \ln W_-(t) = \Psi_+(t) - \Psi_-(t) = \ln g(t)$. Now, from previous Lemma 3.1, it follows on taking the exponent of the asymptotic relation that

$$e^{\Psi(z)} \sim \text{Constant} (z-z_1)^{a_1+ib_1} \text{ as } z \rightarrow z_1$$

$$e^{\Psi(z)} \sim \text{Constant} (z-z_2)^{a_2+ib_2} \text{ as } z \rightarrow z_2$$
Therefore, we choose integers \( n_1, n_2 \) so that \( n_1 + a_1 \in (-1, 0], n_2 + a_2 \in (-1, 0] \). Clearly this choice is unique. 

**Lemma 3.3.** The most general solution to the Riemann-Hilbert problem

\[
\Phi_+ (t) - \Phi_- (t) = \phi(t)
\]

for a smooth simple open curve when \( f \) is Holder continuous, including the end points, is given by

\[
\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s-z} ds + M(z)
\]

where \( M(z) \) is an analytic function everywhere, except for isolated singularities at \( z = z_1 \) and \( z_2 \). If we require \( \Phi(z) \) to have a weak singularity at the end points, i.e. \( |\Phi(z)| \leq C|z - z_{1,2}|^\beta \) for \( \beta > -1 \), then \( M(z) = E(z) \), where \( E \) is an entire function.

**Proof.** Since \( \Phi(z) \) and \( \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s-z} ds \) are analytic everywhere off \( C \), so is \( M(z) \) defined by the expression above. We already know from Lemma 3.1 that \( \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s-z} ds \) has only logarithmic singularity at the end points. So, from the requirement on \( \Phi \), \( M \) can have only weak singularities at end points \( z = z_{1,2} \). However, the RH jump condition implies \( M_+(t) - M_1(t) = 0 \); thus \( M(z) \) is single valued in a neighborhood of each of the end points. Therefore, it has a Laurent series at \( z = z_1 \) and \( z = z_2 \). Since these are supposed to be only weakly singular points, poles or essential singularities at \( z_{1,2} \) are ruled out. This implies \( M(z) = E(z) \) for some entire function \( E(z) \).

**Remark 3.4.** The Theorem above can be proved under weaker assumption that \( \phi \) is Holder continuous except at end points \( z_{1,2} \), where it is weakly singular, i.e. \( |\phi(t)| \leq B|t - z_{1,2}|^\beta \) for \( \beta > -1 \). This will be an exercise.

The generalized Riemann Hilbert (RH) Problem consists of finding sectionally holomorphic function \( L(z) \) so that on a smooth simple curve \( C \),

\[
L_+ (t) = g(t)L_- (t) + f(t)
\]

where \( g \) and \( f \) are Holder continuous on \( C \), except that \( f \) is allowed to be weakly singular at the end points. Further, \( g \neq 0 \). This can be solved by first finding \( W(z) \) so that \( W_+ (t) = g(t)W_- (t) \). with the requirement that \( W \) is only weakly singular at the end points when it is an open curve. From the representations given in last class for closed curve and the one given in Lemma 3.2 for open curve, it is clear that
\( W_+(t) \neq 0 \) including at the end points. Therefore, the generalized RH problem can be written as
\[
\frac{L_+(t)}{W_+(t)} - \frac{L_-(t)}{W_-(t)} = \frac{f(t)}{W_+(t)}
\]
Since \( f \) is Holder continuous and \( W_+ \) is non zero and regular except at end points, where it is weakly singular (see Lemma 3.2), it follows that \( \phi(t) = \frac{f(t)}{W_+(t)} \) is Holder continuous on \( C \), except at end points where it is weakly singular. Note the weak singularity (at best) of \( W_+ \) at the end points guaranteed by right choice of \( n_1, n_2 \) is useful. Therefore, using Lemma 3.3 for open curves and similar Lemma for closed curves from last week’s notes, \( \Phi(z) \equiv \frac{L(z)}{W(z)} \) is given by
\[
\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s)ds}{s-z} + M(z)
\]
implying
\[
L(z) = W(z)\Phi(z) = W(z) \left\{ \frac{1}{2\pi i} \int_C \frac{\phi(s)ds}{s-z} + M(z) \right\}
\]
If we require \( L(z) \) to have at the worst weak singularity at \( z = z_{1,2} \), then \( M(z) = E(z) \), an entire function, as for a closed contour. Additional condition on behavior at \( \infty \) restricts \( E(z) \).

4. Applications: Solving Integral equations of The Cauchy Kind

Definition 4.1. A linear singular integral equation of the Cauchy-type is an equation for the unknown \( h(z) \) on simple smooth curve \( C \) that satisfies:
\[
a(z) h(z) = \lambda \int_C \frac{h(s)}{s-z} + q(z)
\]
where \( a(z), q(z) \) are known Holder continous functions on \( C \) and \( \lambda \) is some specified constant.

We assume \( a(z), q(z) \) to be Holder continuous on \( C \), including the end points. We will limit the discussion for open curves, since closed curves can similarly be dealt with by using the appropriate RH problem for closed curves. For simplicity of description we restrict \( C \) on the real axis; \( C = [-1, 1] \) without loss of generality since other cases can be handled by change of variable with appropriate shifting and scaling. So, consider
\[
a(x) h(x) = \lambda \int_{-1}^1 \frac{h(s)}{s-x} + q(x)
\]
where \( \lambda > 0 \), w.l.o.g. We will further take \( a(x) \) to be real valued, as is natural in applications where they arise. Define

\[
H(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{h(s)ds}{s - z}
\]

for \( z \) off the real interval (-1, 1). Recall Plemelj-formulae,

\[
H_{\pm}(x) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{h(s)ds}{s - x} \pm \frac{h(x)}{2}
\]

So, by using the above, (4.8) may be written as

(4.9) \[ a(x) - \lambda \pi i \] \( H_{+}(x) = [a(x) + \lambda \pi i] H_{-}(x) + q(x) \]

(4.10) \[ H_{+}(x) = \frac{a(x) + \lambda \pi i}{a(x) - \lambda \pi i} H_{-}(x) + \frac{q(x)}{a(x) - \lambda \pi i} \]

Equation (4.10) is the generalized R-H problem we talked about before with \( g(x) = \frac{a(x) + \lambda \pi i}{a(x) - \lambda \pi i} \) and \( f(x) = \frac{q(x)}{a(x) - \lambda \pi i} \). We can proceed as before and solve for \( H(z) \). Note that at the end, we have to find \( h(x) \), which is \( h(x) = H_{+}(x) - H_{-}(x) \).

5. Integral equations with Algebraic Kernels:

Consider an integral equation of the Abel type:

(5.11) \[ \int_{0}^{x} \frac{f(t) dt}{(x-t)^{\alpha}} = g(t) \text{ for } x > 0 \]

where \( \alpha \in (0, 1) \), \( g \) is specified and the unknown is \( f \).

**Definition 5.1.** For \( f \in L^{1}_{\text{loc}}(\mathbb{R}^{+}) \), locally integrable function on the positive real axis, which is exponentially bounded, i.e. \( |f(t)| \leq Be^{ct} \) for some \( c \geq 0 \), we define Laplace-Transform

\[
F(s) = \mathcal{L}[f(.)](s) = \int_{0}^{\infty} e^{-st} f(t) \, dt
\]

**Lemma 5.2.** \( F \) is an analytic function in the domain \( \text{Re } s > c \).

**Proof.** This is left as an exercise. \[ \square \]

**Definition 5.3.** A Laplace convolution of the function between two locally integrable functions \( f \) and \( g \) on \( \mathbb{R}^{+} \), denoted by \( f \ast g \) is defined as:

\[
f \ast g(t) = \int_{0}^{t} d\tau \ f(\tau) \ g(t-\tau) = \int_{0}^{t} g(\tau) \ f(t-\tau) \ d\tau
\]
Lemma 5.4. Assume for $c > 0$ $f' \in L^1(e^{-ct}dt)$ and i.e. locally integrable function that is exponentially bounded with exponent $c$. Then,

$$\mathcal{L}[f'](s) = sF(s) - f(0^+)$$

Proof. Since $|f(t)| \leq C + \int_0^t |f'(\tau)|d\tau$, it is easily seen that $f \in L^1(e^{-ct}dt)$ and that $f$ is bounded at each finite point $t \in \mathbb{R}^+$ with $\lim_{t \to \infty} e^{-c_1t} |f(t)| = 0$ for any $c_1 > c$. Therefore, for $Re s > c$

$$\int_0^\infty e^{-st} f'ds = \int_0^\infty f(t)de^{-st} = -f(0^+) + s \int_0^\infty e^{-st} f(s)ds$$

from which the Lemma follows.

Lemma 5.5. If $f, g \in L^1(e^{-ct}dt)$, i.e. locally integrable functions that are exponentially bounded, then so is $f \ast g$ and

$$\mathcal{L}[f \ast g] = \mathcal{L}[f]\mathcal{L}[g].$$

Proof. We note that using the fact that $f$ and $g$ are locally integrable with respect to $t$ and $\tau$ and using Fubini’s theorem:

$$\int_0^R e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau \leq \int_0^R e^{-st} \left\{ \int_0^t f(\tau)e^{-s(t-\tau)} \right\} d\tau$$

$$= \int_0^R e^{-st} d\tau \left\{ \int_\tau^R g(t-\tau)e^{-s(t-\tau)} \right\}$$

Since each of the integrals exist as $R \to \infty$, for $Re s > c$ by introducing new variable $\sigma = t - \tau$, we obtain

$$\int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^\infty e^{-st} d\tau \left\{ \int_0^\infty g(\sigma)e^{-s\sigma}d\sigma \right\},$$

which gives the desired result.

Remark 5.6. The inverse Laplace transform, denoted by $\mathcal{L}^{-1}$ is unique, when it exists. This follows from an appropriate extension of Fourier-inversion Theorem. $\mathcal{L}^{-1}$ has an inverse Laplace-Transform representation as will be seen later.

Lemma 5.7. Assume $g \in L^1(e^{-ct}dt)$ The solution of Abel’s integral equation is given by

$$f(x) = \frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} \frac{d}{dx} \int_0^x g(t) (x-t)^{\alpha-1} dt$$

where for $Re \alpha > 0$,

$$\Gamma(\alpha) = \int_0^\infty e^{-p} p^{\alpha-1} dp$$
Proof. We look for solution $f \in L^1(e^{-ct}dt)$. From Laplace transform of \((5.11)\) and using Lemma 5.5, we obtain

$$F(s)L\{t^{-\alpha}\}(s) = G(s) \tag{8}$$

where $F(s) = L[f(t)](s)$ and $G(s) = L[g(t)](s)$. However, $L[t^{-\alpha}](s) = s^{\alpha-1} \Gamma (1 - \alpha)$. Therefore

$$(5.13) \quad F(s) = \frac{s}{\Gamma(1-\alpha)} s^{-\alpha} G(s)$$

Applying $L^{-1}$ to the above and using Lemma 5.4 we obtain

$$f(x) = \frac{d}{dx} \left\{ L^{-1} \left[ \frac{1}{\Gamma(1-\alpha)} s^{-\alpha} G(s) \right] (x) \right\}$$

Since

$$L \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right] (s) = s^{-\alpha},$$

it follows that

$$L^{-1} [s^{-\alpha}] (t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

Therefore, from \((5.13)\) and convolution result Lemma 5.5

$$f(x) = \frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} \frac{d}{dx} \left\{ \int_0^x dt \ g(t) \ (x-t)^{\alpha-1} \right\}$$

Remark 5.8. It turns out, from using representation of $\Gamma$ function, one can show that

$$\frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} = \frac{\sin \alpha \pi}{\pi}$$

Remark 5.9. The result in Lemma 5.7 can be combined with Riemann Hilbert techniques in solving the following type of integral equation with algebraic Kernel.

Example: Determine $f$ so that

$$\int_0^1 \frac{f(t)dt}{|x-t|^\alpha} = g(x) \tag{5.14}$$

for $x$ in $(0,1)$, where $\alpha \in (0,1)$. Define

$$F(z) = \int_0^1 \frac{f(t) \ dt}{(z-t)^\alpha} \tag{5.15}$$
In interpreting the integrand, we take the principal branch of \((z - t)^\alpha\), i.e. \(\text{arg} (z-t)\) in the interval \((-\pi, \pi]\). It will be shortly seen that \(F(z)\), as defined above, has a branch cut between \((-\infty, 1)\). For \(x \in (0, 1)\),

\[(5.16)\]
\[F_+(x) = \int_0^x \frac{f(t)dt}{(x-t)^\alpha} + e^{-i\alpha\pi} \int_x^1 \frac{f(t)dt}{(t-x)^\alpha} \]

\[(5.17)\]
\[F_-(x) = \int_0^x \frac{f(t)dt}{(x-t)^\alpha} + e^{i\alpha\pi} \int_x^1 \frac{f(t)dt}{(t-x)^\alpha} \]

Note that from \((5.16)\) and \((5.17)\),

\[(5.18)\]
\[e^{i\alpha \pi} F_+(x) - e^{-i\alpha\pi} F_-(x) = 2i \sin \alpha \pi \int_0^x \frac{f(t)dt}{(x-t)^\alpha} \]

\[(5.19)\]
\[F_+(x) - F_-(x) = -2i \sin \alpha \pi \int_x^1 \frac{f(t)dt}{(t-x)^\alpha} \]

Using \((5.18)\) and \((5.19)\), equation \((5.14)\) can be written as:

\[(5.20)\]
\[F_+(x) = -e^{-i\alpha \pi} F_-(x) + (1 + e^{-i\alpha \pi}) g(x) \]

for \(x \in (0, 1)\). For \(x \in (-\infty, 0)\),

\[(5.21)\]
\[F_+(x) = e^{-2i\alpha \pi} F_-(x) \]

This is again a Riemann Hilbert problem. However, our previous method for solving the Riemann Hilbert problem is not applicable here, because the integrals arising in this case does not exist, because of the infinite length of contour \(C\). However, since coefficients in \((5.20)\) and \((5.21)\) are merely constants, a simple change of variable

\[(5.22)\]
\[W(z) = z^\beta (z-1)^\gamma F(z) \]

transforms it to a simpler RH problem with proper choice of \(\beta\) and \(\gamma\). We seek \(\beta\) and \(\gamma\) so that \((5.21)\) translates to \(W_+(x) = W_-(x)\), while \((5.21)\) translates into \(W_+(x) = W_-(x) + \phi(x)\) for some \(\phi\). We can check that this is possible with \(\beta = \gamma = \frac{1}{2}(\alpha - 1)\) is such a choice, i.e.

\[(5.23)\]
\[W(z) = z^{(\alpha-1)/2} (z-1)^{(\alpha-1)/2} F(z) \]

In that case \(W(z)\) is only discontinuous for \(z\) on the real axis between \((0, 1)\); there is no branch cut any more from \((-\infty, 0)\). Further, \((5.21)\) becomes:

\[(5.24)\]
\[W_+(x) - W_-(x) = -2i \cos \alpha \pi/2 x^{(\alpha-1)/2} (1-x)^{(\alpha-1)/2} g(x) \]
From the observation that (??) and (5.23) imply that $W(z) \to 0$ at $\infty$, therefore

\[(5.25) \quad W(z) = -\frac{\cos \alpha \pi / 2}{\pi} \int_{0}^{1} \frac{dt}{t - z} t^{(\alpha - 1)/2} (1 - t)^{(\alpha - 1)/2} g(t)\]

On the real axis, in $(0, 1)$, we have

\[(5.26) \quad x^{(\alpha - 1)/2} (1 - x)^{(\alpha - 1)/2} e^{i\pi(\alpha - 1)/2} F_+(x) = W_+(x)\]

\[= -\frac{\cos \alpha \pi / 2}{\pi} \int_{0}^{1} \frac{dt}{t - x} t^{(\alpha - 1)/2} (1 - t)^{(\alpha - 1)/2} g(t) - i \cos [\alpha \pi / 2] x^{(\alpha - 1)/2} (1 - x)^{(1 - \alpha)/2} g(x)\]

\[(5.27) \quad x^{(\alpha - 1)/2} (1 - x)^{(\alpha - 1)/2} e^{-i\pi(\alpha - 1)/2} F_-(x) = W_-(x)\]

\[= -\frac{\cos \alpha \pi / 2}{\pi} \int_{0}^{1} \frac{dt}{t - x} t^{(\alpha - 1)/2} (1 - t)^{(\alpha - 1)/2} g(t) + i \cos [\alpha \pi / 2] x^{(\alpha - 1)/2} (1 - x)^{(1 - \alpha)/2} g(x)\]

Using Lemma 5.7, (5.18), (5.26) and (5.27), we obtain

\[(5.28) \quad f(x) = \frac{\sin \alpha \pi}{2\pi} \frac{d}{dx} \left\{ \int_{0}^{x} \frac{g(t)}{(x - t)^{1-\alpha}} dt \right\} - \frac{\cos^2(\alpha \pi / 2)}{\pi^2} \frac{d}{dx} \left[ \int_{0}^{x} dt \left\{ \frac{t(1 - t)^{(1-\alpha)/2}}{(x - t)^{1-\alpha}} \left\{ \int_{0}^{1} \frac{g(\tau)[\tau(1 - \tau)]^{(\alpha - 1)/2}}{\tau - t} d\tau \right\} d\tau \right\} dt \right]