Lecture 15, Math 805
Elliptic functions

Definition 1.1: A function $f(z)$ of a complex variable $z$ is called a doubly periodic if there exists two nonzero numbers $\omega_1$ and $\omega_2$, whose ratio is not purely real, such that

$$f(z + 2 \omega_1) = f(z) \quad ; \quad f(z + 2 \omega_2) = f(z)$$

(1)

for all values of $z$ for which $f(z)$ exists.

Definition 1.2: A doubly-periodic function which is analytic (except at poles), and has no singularities other than poles in a finite part of the complex plane, is called an elliptic function.

Remark: The periods $2 \omega_1$ and $2 \omega_2$ play the same role in the elliptic function theory as a single period does in theory of circular (trigonometric) functions.

Definition 1.3: If inside or on the parallelogram obtained by joining the points 0, $2 \omega_1$, $2 \omega_2 + 2 \omega_1$, $2 \omega_2$ and 0, there is no point $\omega$, except for the vertices, with the property that $f(z + \omega) = f(z)$, then this parallelogram is called the fundamental period-parallelogram.

![Figure 1: Fundamental Period Parallelogram](image)

Remark: It is clear that the $z$-plane can be covered by a network of period parallelograms or mesh, that are translations of the fundamental period-parallelogram (in Fig. 1) by $2 m \omega_1 + 2 n \omega_2$, $m$ and $n$ being integers. From the doubly periodic property, the elliptic function takes on the same set of values in each period parallelogram.

Remark: For purposes of integration, it is not convenient to deal with the actual meshes if they have singularities on the function on the boundaries of the period parallelogram; because of periodicity, there is no loss of generality in choosing a contour, that does not coincide with a period parallelogram, but is a mere translation in such a way that there are
no singularities of the function on the boundary. Such a parallelogram will be referred to as a cell.

**Definition 1.4:** The set of poles (or zeros) of an elliptic function in any given cell is called an irreducible set; all other poles (or zeros) of the function are congruent to it, i.e. differ by an additive factor of $2\ m\ \omega_1 + 2\ n\ \omega_2$ for some integers $m$ and $n$.

**Lemma 1.1:** The number of poles of an elliptic function $f(z)$ in any cell must be finite.

**Proof:** If not, there must be a limit point within the cell. The limit point is a non-isolated singularity of $f(z)$, which contradicts the definition of elliptic function.

**Lemma 1.2:** The number of zeros of an elliptic function $f(z)$ in any cell must be finite as well.

**Proof:** Note that reciprocal $f(z)$ is an elliptic function as well, since analytic functions can have only isolated zeros (recall Math 804 Theorem). Applying Lemma 1.1 to $1/f(z)$, Lemma 1.2 follows.

**Lemma 1.3:** The sum of residues of an elliptic function, $f(z)$, at its poles in any cell is zero.

**Proof:** Let $C$ be the contour coinciding with the boundaries of a cell, whose vertices are at $t, \ t + 2\omega_1, \ t + 2\omega_1 + 2\omega_2$ and $t + 2\omega_2$. Then

$$\frac{1}{2\pi i} \oint_C f(z) \, dz = \frac{1}{2\pi i} \left\{ \int_t^{t + 2\omega_1} + \int_{t + 2\omega_1}^{t + 2\omega_1 + 2\omega_2} + \int_{t + 2\omega_1 + 2\omega_2}^{t + 2\omega_2} + \int_{t + 2\omega_2}^t \right\} f(z) \, dz$$

(2)

The first and third integrals, as well as second and fourth integrals, can be combined as:

$$\frac{1}{2\pi i} \int_t^{t + 2\omega_1} [f(z) - f(z + 2\omega_2)] \, dz - \frac{1}{2\pi i} \int_{t + 2\omega_2}^{t + 2\omega_2} [f(z) - f(z + 2\omega_1)] \, dz$$

and each of these integrands vanish due to periodicity.

**Remark:** There cannot be an elliptic function with only one simple pole. This follows from the fact that it has only one residue which must be zero.

**Lemma 1.4:** An elliptic function, $f(z)$, with no poles in a cell is merely a constant.

**Proof:** If $f(z)$ is analytic everywhere in a cell, it follows $|f(z)| \leq K$ (constant) for $z$ on or within the cell, since the cell is a compact set. From periodicity, $f(z)$ is bounded everywhere. From Liouville’s theorem, $f(z)$ must be a constant.

**Definition 1.5:** The order of an elliptic function The order of an elliptic function is the number of roots of the equation

$$f(z) = c$$

(3)
which lie within a cell. As will be shown, this is independent of the value of \( c \) chosen, but only depends on the elliptic function under consideration.

**Lemma 1.5:** The order of an elliptic function \( f(z) \) is the same as the number of poles (including multiplicities) of \( f(z) \) in a cell.

**Proof:** Consider

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z) - c} \, dz
\]  

(4)

Without loss of generality, the cell boundary \( C \) is chosen so that it not only avoids poles of \( f(z) \), but also zeros of \( f(z) - c \). Breaking up the integral (4) into four parts, as in the proof of Lemma 1.3 and using periodicity of the integrand as before, we conclude that the expression in (4) is zero. However, (4) is the merely the difference between the number of zeros of \( f(z) - c \) and the number of poles of \( f(z) - c \) (the same as \( f(z) \)), within the cell. Thus, the Lemma follows.

**Remark:** Note that the number of poles of \( f(z) \) is not dependent on \( c \) and hence the order of an elliptic function, as defined in definition 1.5, is independent of \( c \), as claimed above.

**Lemma 1.6:** The sum of the irreducible zeros (including multiplicity) is congruent to the sum of poles (including multiplicity).

**Proof:** Let \( C \) denote the boundary of a cell, that will be chosen, without loss of generality, to avoid any zeros or poles of \( f(z) \). Then,

\[
\frac{1}{2\pi i} \oint_C \frac{zf'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \left\{ \int_t^{t+2\omega_1} f'(z) - f(z) \, dz \right. \\
- \left. \int_t^{t+2\omega_2} f'(z) - f(z) \, dz \right. \\
+ \left. \int_t^{t+2\omega_1} f'(z) - f(z) \, dz \right. \\
- \left. \int_t^{t+2\omega_2} f'(z) - f(z) \, dz \right. \\
= \frac{1}{2\pi i} \left\{ -2 \omega_2 \int_t^{t+2\omega_1} f'(z) - f(z) \, dz \right. \\
+ \left. 2 \omega_1 \int_t^{t+2\omega_2} f'(z) - f(z) \, dz \right. \\
= \frac{1}{2\pi i} \left\{ -2 \omega_2 \int_t^{t+2\omega_1} f'(z) - f(z) \, dz \right. \\
+ \left. 2 \omega_1 \int_t^{t+2\omega_2} f'(z) - f(z) \, dz \right. \\
= \frac{1}{2\pi i} \left\{ -2 \omega_2 \left[ \ln f(z) \right]_t^{t+2\omega_1} + 2 \omega_1 \left[ \ln f(z) \right]_t^{t+2\omega_2} \right. \\
= \frac{1}{2} \omega_2 \left[ \ln f(z) \right]_t^{t+2\omega_1} + 2 \omega_1 \left[ \ln f(z) \right]_t^{t+2\omega_2} \right) = 2 m \omega_1 + 2 n \omega_2
\]  

(5)

However, we note that \( z \) \( f'(z)/f(z) \) has a simple pole with residue \( z_0 \) at a simple zero \( z_0 \) (\( z_0 \) being the location of the zero) and has a simple pole with residue \(-z_p \) at a simple pole \( z_p \). For an higher order zero or poles, the residues are multiplied by the order of zero or pole. Thus, from residue theory, and relation (5), it follows that

\[
\sum_j z_{0j} - \sum_k z_{pk} = 2 m \omega_1 + 2 n \omega_2
\]
where the multiplicity of zeros or poles are included in the summation (i.e. if there is a second order zero at a point, then two of the indices $j$ in the summation correspond to that point). Hence the Lemma follows.
Lecture 16, Math 806

Construction of Weirstrass elliptic function \( \mathcal{P}(z) \)

Definition: We define the Weirstrass elliptic function as:

\[
\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n} \left\{ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\}
\]  

(1)

where

\[
\Omega_{m,n} = 2m\omega_1 + 2n\omega_2
\]  

(2)

and the summation in (1) extends to all integer values (positive, negative or zero) of \( m \) and \( n \), except when \( m \) and \( n \) are both zero.

Remark: When \( |\Omega_{m,n}| \) is large, it is clear that the general term of the series in (1) is \( O(|\Omega_{m,n}|^{-3}) \), and so the double sum over \( m \) and \( n \) converges absolutely and uniformly in any compact set that excludes the set of points \( \Omega_{m,n} \). Thus \( \mathcal{P}(z) \) as defined above is indeed an analytic function, except for double poles at \( \Omega_{m,n} \).

Remark: To show periodicity and other properties of \( \mathcal{P}(z) \), it is easier to first consider the properties of \( \mathcal{P}'(z) \). Because of uniform convergence, it follows that

\[
\mathcal{P}'(z) = -2 \sum_{m,n} \frac{1}{(z - \Omega_{m,n})^3}
\]  

(3)

On rearranging the shifting the index \( m \) in the above summation, it is clear that

\[
\mathcal{P}'(z + 2\omega_1) = \mathcal{P}'(z)
\]  

(4)

Similarly, \( 2\omega_1 \) in the above, can be replaced by \( 2\omega_2 \). Thus, it is clear that \( \mathcal{P}'(z) \) is a doubly periodic analytic function (elliptic function) with periods \( 2\omega_1 \) and \( 2\omega_2 \), and order three. Further,

\[
\mathcal{P}'(-z) = 2 \sum_{m,n} \frac{1}{(z + \Omega_{m,n})^3}
\]  

(5)

However, we can replace \( \Omega_{m,n} \) in the above by \( -\Omega_{m,n} \), since the set of values attained by \( -\Omega_{m,n} \) is the same. Therefore

\[
\mathcal{P}'(-z) = 2 \sum_{m,n} \frac{1}{(z - \Omega_{m,n})^3} = -\mathcal{P}(z)
\]  

(6)

In the same manner, it follows that

\[
\mathcal{P}(-z) = \frac{1}{z^2} + \sum_{m,n} \left\{ \frac{1}{(z + \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\} = \mathcal{P}(z)
\]  

(7)
On integrating (4), it follows that

\[ \mathcal{P}(z + 2 \omega_1) = \mathcal{P}(z) + A \]  

for some constant \( A \). However, if we now substitute \( z = - \omega_1 \) into (7), we obtain

\[ \mathcal{P}(-\omega_1) = \mathcal{P}(\omega_1) + A \]

Using (8), it follows that \( A = 0 \). Hence, from (8), we obtain

\[ \mathcal{P}(z + 2\omega_1) = \mathcal{P}(z) \]  

(9)

Similarly

\[ \mathcal{P}(z + 2\omega_2) = \mathcal{P}(z) \]  

(10)

Thus \( \mathcal{P}(z) \) is a second order elliptic function, with fundamental periods \( 2\omega_1 \) and \( 2\omega_2 \). Notice it cannot possibly have a smaller period, because \( \mathcal{P}(z) \) has a pole at \( z = 0 \) and there are no other poles in the period parallelogram, with vertices \( 0, 2\omega_1, 2\omega_1 + 2\omega_2 \) and \( 2\omega_2 \).

**Lemma 2.1:** \( \mathcal{P}(z) \) satisfies the differential equation:

\[ \mathcal{P}''(z) = 4\mathcal{P}^2(z) - g_2 \mathcal{P}(z) - g_3 \]

(11)

where

\[ g_2 = 60 \sum_{m,n} \Omega_{m,n}^{-4}, \quad g_3 = 140 \sum_{m,n} \Omega_{m,n}^{-6} \]

(12)

**Proof:** We first note that \( \mathcal{P}(z) - z^{-2} \) is a regular function in a neighborhood of \( z = 0 \) and has the Taylor series expansion:

\[ \mathcal{P}(z) - z^{-2} = \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6) \]

(13)

where \( g_2 \) and \( g_3 \) are as defined by (12). Further on differentiating, it follows that

\[ \mathcal{P}'(z) = -2 z^{-3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + O(z^5) \]

(14)

Cubing and squaring relations (13) and (14), respectively, we get

\[ \mathcal{P}^3(z) = z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + O(z^2) \]

(15)

\[ \mathcal{P}^2(z) = 4 z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + O(z^2) \]

(16)

Hence

\[ \mathcal{P}''(z) - 4 \mathcal{P}^3(z) = -g_2 z^{-2} - g_3 + O(z^2) \]

(17)
Therefore, in a neighborhood of $z = 0$,

$$\mathcal{P}^2(z) - 4 \mathcal{P}^3(z) + g_2 \mathcal{P}(z) + g_3 = O(z^2) \quad (18)$$

Since, the expression on the left hand side of (18) is clearly an elliptic function with possible singularities only at singularities of $\mathcal{P}$ and $\mathcal{P}'$, i.e at $z = 0$ and all points congruent to it, it follows from (18) that it has no singularities anywhere (since it cannot have one at 0). Thus, the left hand side of (18) must be a constant as it is entire and bounded. Equation (18) implies that the constant must be zero, since the right hand side is 0 at $z = 0$. Thus, (11) follows and the lemma is proved.

**Remark:** Conversely, given a differential equation in the form:

$$\left( \frac{dy}{dz} \right)^2 = 4 y^3 - g_2 y - g_3 \quad (19)$$

with given $g_2$ and $g_3$, if numbers $\omega_1$ and $\omega_2$ can be determined so that (12) is valid, then the general solution of the differential equation is given by

$$y(z) = \mathcal{P}(\pm z + \alpha) \quad (20)$$

where $\alpha$ is some constant of integration. This can be proved by simply transforming the dependent variable $y$ through the transformation $y = \mathcal{P}(u)$. It is then seen that the differential equation (19) becomes

$$\left( \frac{du}{dz} \right)^2 = 1$$

**Remark:** The differential equation (19) arises naturally in studying the travelling wave solution to the Korteweg Devries equation for $u(x, t)$ of the form:

$$u_t + 6u u_x - u_{xxx} = 0 \quad (21)$$

For a travelling wave solution $u(x, t) = F(x - c t)$. Then

$$-c F' + 6 F F' - F'' = 0 \quad (22)$$

Integrating once,

$$-c F + 3 F^2 - F''' = d \quad (23)$$

for some constant $d$. Multiplying (23) by $2 F'$ and integrating again, we get

$$F'^2 = -c F^2 + 2 F^3 - 2 d F + b \quad (24)$$
for some constant $b$. By shifting $F$ by a constant, i.e. introducing $F = G + \text{constant}$ and choosing constant suitably, we get an equation for $G$ of the form

$$G^2 = 2 \, G^3 - g_2 \, G - g_3 \quad (24)$$

This can be further transformed into (20) by merely scaling both the dependent and independent variables by constants. Thus, the elliptic function is useful in studying this integrable partial differential equation.
Lecture 17, Math 805

Integral formula for \( \mathcal{P}(z) \)

**Lemma 3.1:** If introduce

\[
 z = h(\zeta) = \int_{\zeta}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} \, dt ,
\]

where the path of integration may be any curve which does not pass through a zero of \( 4t^3 - g_2 t - g_3 \), then \( h(\mathcal{P}(z)) = z \).

**Proof:** On differentiating (1), we obtain

\[
 \left( \frac{d\zeta}{dz} \right)^2 = 4 \zeta^3 - 2g_2 \zeta - g_3
\]

so that

\[
 \zeta = \mathcal{P}(\pm z + \alpha) = \mathcal{P}(z \pm \alpha)
\]

Now, in (1), if we let \( \zeta \to \infty \), we obtain \( z \to 0 \). This means that \( \mathcal{P}(z \pm \alpha) \) blows up at \( z = 0 \). This means that \( \alpha \) is either 0 or a point congruent to it. Thus \( \zeta = \mathcal{P}(z) \) and the Lemma is proved.

**Lemma 3.2:** The elliptic function \( \mathcal{P} \) satisfies the following addition theorem:

\[
 \det \begin{pmatrix} \mathcal{P}(z) & \mathcal{P}'(z) & 1 \\ \mathcal{P}(y) & \mathcal{P}'(y) & 1 \\ \mathcal{P}(z+y) & -\mathcal{P}'(z+y) & 1 \end{pmatrix} = 0
\]  

**Proof:** Consider the following set of two linear equations for determining \( A \) and \( B \):

\[
 \mathcal{P}'(z) = A \mathcal{P}(z) + B \quad (5)
\]

\[
 \mathcal{P}'(y) = A \mathcal{P}(y) + B \quad (6)
\]

These determine \( A \) and \( B \) uniquely in terms of \( z \) and \( y \), unless \( \mathcal{P}(z) = \mathcal{P}(y) \), i.e. unless \( z = \pm y \ (\text{mod} \ 2\omega_1, 2\omega_2) \). For \( A \) and \( B \) as determined above, consider

\[
 \mathcal{P}'(\zeta) - A \mathcal{P}(\zeta) - B
\]

It has a triple pole at \( \zeta = 0 \) and consequently three irreducible zeros (i.e. three zeros within a cell). Two of these zeros are clearly \( \zeta = z \) and \( \zeta = y \) and the third irreducible zero must be congruent to \( -z - y \) (recall that the sum of pole location within a cell, which is zero in this case, is congruent to the sum of zero locations). Thus,

\[
 \mathcal{P}'(-z - y) = A \mathcal{P}(-z - y) + B
\]
which implies
\[-\mathcal{P}'(z + y) = A \mathcal{P}(z + y) + B\]  
(8)

Since (5), (6) and (8) can be viewed as three equations for two unknowns \(A\) and \(B\), their consistency conditions require that (4) is satisfied.

**Remark:** A more symmetrical form of (4) is that

\[
\text{det} \begin{pmatrix} \mathcal{P}(u) & \mathcal{P}'(u) & 1 \\ \mathcal{P}(v) & \mathcal{P}'(v) & 1 \\ \mathcal{P}(w) & \mathcal{P}'(w) & 1 \end{pmatrix} = 0
\]  
(9)

for \(u + v + w = 0\).

**Lemma 3.3:** Another addition theorem for the Weierstrass elliptic function is that:

\[
\mathcal{P}(z + y) = \frac{1}{4} \left\{ \mathcal{P}''(z) - \mathcal{P}'(y) \right\}^2 - \mathcal{P}(z) - \mathcal{P}(y)
\]  
(10)

**Proof:** Retaining the same notation as in Lemma 3.2, we see that the values of \(\zeta\) for which \(\mathcal{P}'(\zeta) - A \mathcal{P}(\zeta) - B\) vanish, are congruent to one of the points \(z, y, -z - y\). Hence \(\mathcal{P}''(z) - \{A \mathcal{P}(z) + B\}^2\) vanishes when \(\zeta\) is congruent to the points \(z, y, -z - y\). Recalling the differential equation satisfied by \(\mathcal{P}\), this means that

\[
4 \mathcal{P}^3(\zeta) - A^2 \mathcal{P}^2(\zeta) - (2AB + g_2)\mathcal{P}(\zeta) - (B^2 + g_3)
\]  
(11)

vanishes when \(\mathcal{P}(\zeta)\) is equal to any one of \(\mathcal{P}(z), \mathcal{P}(y), \mathcal{P}(z + y)\). For general values of \(z\) and \(y\), \(\mathcal{P}(z), \mathcal{P}(y)\) and \(\mathcal{P}(z + y)\) are unequal and so they are all the roots of

\[
4Z^3 - A^2 Z^2 - (2AB + g_2) Z - (B^2 + g_3) = 0
\]  
(12)

Consequently, from the relation of the zeros of a cubic to its coefficients, it follows that

\[
\mathcal{P}(z) + \mathcal{P}(y) + \mathcal{P}(z + y) = \frac{1}{4} A^2
\]  
(13)

However, from solving for \(A\) and \(B\) in (5) and (6), and using the value of \(A\), we obtain the relation (10) and the theorem is proved.

**Corollary 3.1:** The duplication formulae is given by

\[
\mathcal{P}(2z) = \frac{1}{4} \left\{ \frac{\mathcal{P}''(z)}{\mathcal{P}'(z)} \right\}^2 - 2 \mathcal{P}(z)
\]  
(14)

**Proof:** Consider the formula (10), with \(y \to z\), then using L’Hopital’s rule, (14) follows.

**Comment:** Differentiating

\[
\mathcal{P}''(z) = 4 \mathcal{P}^3(z) - g_2 \mathcal{P}(z) - g_3
\]  
(15)
we obtain on cancellation of $\mathcal{P}'(z)$,

$$2 \mathcal{P}''(z) = 12 \mathcal{P}^2(z) - g_2$$

(16)

Using (15) and (16) in the relation (14), it is possible to express $\mathcal{P}(2z)$ in terms of $\mathcal{P}(z)$ alone, though one ought to be careful in the proper interpretation of the square-root.

**Relations satisfied by $e_1$, $e_2$ and $e_3$:**

**Definition 3.1:** Define $e_1 = \mathcal{P}(\omega_1), e_2 = \mathcal{P}(\omega_2)$ and $e_3 = \mathcal{P}(\omega_3)$, where $\omega_3 = -\omega_1 - \omega_2$.

**Lemma 3.5:** $e_1$, $e_2$ and $e_3$ are all unequal to each other and are distinct roots of

$$4 t^3 - g_2 t - g_3$$

(17)

**Proof:** First, we note that

$$\mathcal{P}'(-\omega_1) = -\mathcal{P}'(\omega_1) = -\mathcal{P}'(2 \omega_1 - \omega_1) = -\mathcal{P}'(\omega_1)$$

(18)

Hence $\mathcal{P}'(\omega_1) = 0$. The same is true for at $\omega_2$ and $\omega_3$. Since $\mathcal{P}'(z)$ is an elliptic function whose only singularities are triple poles at points congruent to the origin, $\mathcal{P}'(z)$ has three and only three irreducible zeros. Therefore the only zeros of $\mathcal{P}'(z)$ are at points congruent to $\omega_1, \omega_2$ and $\omega_3$.

Next, consider $\mathcal{P}(z) - e_1$. This vanishes at $\omega_1$ and since $\mathcal{P}'(\omega_1) = 0$, it has a double zero at $\omega_1$. Since $\mathcal{P}(z)$ is a second order elliptic function, the only zeros of $\mathcal{P}(z) - e_1$ is at $\omega_1$. In the same manner, the only irreducible zero of $\mathcal{P}(z) - e_2$ is at $\omega_2$ and the only irreducible zero of $\mathcal{P}(z) - e_3$ at $\omega_3$. It follows therefore that $e_1 \neq e_2 \neq e_3$, as otherwise one of $\mathcal{P}(z) - e_j$ would have a zero at more than one point. Also, from the differential equation for $\mathcal{P}(z)$ and the vanishing of $\mathcal{P}'$ at $\omega_j$, it follows that

$$4 \mathcal{P}^3(\omega_j) - g_2 \mathcal{P}(\omega_j) - g_3 = 0$$

(19)

Thus, each $e_j$ is a distinct root of (17) and the lemma is proved.

**Comment:** From the formulae connecting roots of equation (17) with their coefficients, it follows that

$$e_1 + e_2 + e_3 = 0$$

(20)

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{1}{4} g_2$$

(21)

$$e_1 e_2 e_3 = \frac{1}{4} g_3$$

(22)

Once $e_1, e_2$ and $e_3$ are found in terms of $g_2$ and $g_3$, expressions for $\omega_1, \omega_2$ and $\omega_3$ follow from the inverse relation given in Lemma 3.1.
**Exercise 1:** Suppose $g_2$ and $g_3$ are real and the discriminant $g_2^3 - 27 g_3^2 > 0$. Show that $e_j$ are all real; choosing them so that $e_1 > e_2 > e_3$, show that

$$\omega_1 = \int_{e_1}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} \, dt$$

(23)

$$\omega_3 = -i \int_{-\infty}^{e_3} (4t^3 - g_2 t - g_3)^{-1/2} \, dt$$

(24)

so that $\omega_1$ is real and $\omega_3$ is purely imaginary.

**Exercise 2:** Show that, in the circumstances of Exercise 1, $\mathcal{P}(z)$ is real on the perimeter of the rectangle whose corners are 0, $\omega_3$, $\omega_1 + \omega_3$ and $\omega_1$. Determine that $\mathcal{P}(z)$ conformally maps this rectangle to the upper-half plane. We know that the upper-half plane can be mapped to the interior of the rectangle given through Schwartz Christoffel transformation. Relate this formulae to that given as the inverse of $\mathcal{P}(z)$ in Lemma 3.1.

**Lemma 3.6:** Expression for the addition of a half-period to the argument of $\mathcal{P}(z)$ is given as:

$$\mathcal{P}(z + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\mathcal{P}(z) - e_1}$$

(25)

**Proof:** Using (10), we get

$$\mathcal{P}(z + \omega_1) + \mathcal{P}(z) + \mathcal{P}(\omega_1) = \frac{1}{4} \left\{ \frac{\mathcal{P}'(z) - \mathcal{P}'(\omega_1)}{\mathcal{P}(z) - \mathcal{P}(\omega_1)} \right\}^2$$

(26)

Since

$$\mathcal{P}^2(z) = 4 \prod_{r=1}^{3} [P(z) - e_r]$$

(27)

we have

$$\mathcal{P}(z + \omega_1) = \frac{[\mathcal{P}(z) - e_2][\mathcal{P}(z) - e_3]}{\mathcal{P}(z) - e_1} - \mathcal{P}(z) - e_1$$

(28)

This results in (25) and the Lemma is proved.
Lecture 18, Math 805
More on elliptic function

Lemma 4.1: An arbitrary even elliptic function \( \phi(z) \), with irreducible zeros and poles at \( \pm a_j \) and \( \pm b_j \) respectively, with \( j = 1, 2, \ldots, n \) can be written as

\[
\phi(z) = A_1 \prod_{r=1}^{n} \left\{ \frac{\mathcal{P}(z) - \mathcal{P}(a_r)}{\mathcal{P}(z) - \mathcal{P}(b_r)} \right\}
\]  

(1)

for some constant \( A_1 \). If any of the constants \( a_r \) or \( b_r \) is congruent to the origin, the corresponding factor \( \mathcal{P}(z) - \mathcal{P}(a_r) \) or \( \mathcal{P}(z) - \mathcal{P}(b_r) \) is omitted in the product in (1).

Proof: Consider the function

\[
\frac{1}{\phi(z)} \prod_{r=1}^{n} \left\{ \frac{\mathcal{P}(z) - \mathcal{P}(a_r)}{\mathcal{P}(z) - \mathcal{P}(b_r)} \right\}
\]  

(2)

It is an elliptic function of \( z \). It has no poles since the zeros of \( \phi(z) \) coincide with the zeros of the numerator; while the poles of \( \phi(z) \) coincide with the zeros of the denominator. Thus, from Liouville’s theorem, (2) must be some constant \( A_1 \). Hence (1) follows and the lemma is proved.

Lemma 4.2: Any elliptic function can be expressed in terms of the Weierstrass elliptic function \( \mathcal{P}(z) \) and its derivativ

Proof: We note that any elliptic function \( f(z) \) can decomposed as:

\[
f(z) = [f(z) + f(-z)] + [f(z) - f(-z)] \frac{\mathcal{P}'(z)}{\mathcal{P}'(z)}
\]  

(3)

Each of the terms \( f(z) + f(-z) \) and \( [f(z) - f(-z)]/\mathcal{P}'(z) \) are clearly elliptic functions. Applying Lemma 4.1, each of these terms can be written in the form (1). Thus the Lemma is proved.

Comment: There is an expression connecting two elliptic functions with the same period. This can be obtained as follows: As discussed before, each elliptic function can be expressed in terms of \( \mathcal{P}(z) \) and \( \mathcal{P}'(z) \). Eliminating \( \mathcal{P}(z) \) and \( \mathcal{P}'(z) \) algebraically between these two equations and

\[
\mathcal{P}^2(z) = 4 \mathcal{P}^3(z) - g_2 \mathcal{P}(z) - g_3
\]  

(4)

we can obtain an algebraic relation connecting the two elliptic functions directly.

Comment: One cannot perform the following integration in terms of elementary functions:

\[
\int^x (a_0 x^4 + 4 a_1 x^3 + 6 a_2 x^2 + 4 a_3 x + a_4)^{-1/2} \, dx
\]  

(5)

However, elliptic functions provide an alternate expression for the answer.
Lemma 4.3: Define
\[ f(x) = a_0 x^4 + 4 a_1 x^3 + 6 a_2 x^2 + 4 a_3 x + a_4 \]  
and let
\[ z = \int_{x_0}^{x} \left[ f(t) \right]^{-1/2} \, dt \]
where \( x_0 \) is any root of \( f(x) = 0 \); then it is possible to express \( x \) as a rational function of \( \mathcal{P}(z) \), with invariants \( g_2 \) and \( g_3 \), where
\[ g_2 = a_0 a_4 - 4 a_1 a_3 + 3 a_2^2 \]  
\[ g_3 = a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4 \]  

Proof: By Taylor’s theorem:
\[ f(t) = 4 A_3 (t - x_0) + 6 A_2 (t - x_0)^2 + 4 A_1 (t - x_0)^3 + A_0 (t - x_0)^4 \]
where
\[ A_0 = a_0 \ , \ A_1 = a_0 x_0 + a_1 \ , \ A_2 = a_0 x_0^2 + 2 a_1 x_0 + a_2 \]  
\[ A_3 = a_0 x_0^3 + 3 a_1 x_0^2 + 3 a_2 x_0 + a_3 \]  
On writing \( \tau = (t - x_0)^{-1} \) and \( (x - x_0)^{-1} = \xi \), we have
\[ z = \int_{\xi}^{\infty} \left[ 4 A_3 \tau^3 + 6 A_2 \tau^2 + 4 A_1 \tau + A_0 \right]^{-1/2} \, d\tau \]
We remove the quadratic term in the cubic involved through the transformation
\[ \tau = A_3^{-1} \left( \sigma - \frac{1}{2} A_2 \right) \]  
\[ \xi = A_3^{-1} (s - \frac{1}{2} A_2) \]
and we get
\[ z = \int_{s}^{\infty} \left\{ 4 \sigma^3 - g_2 \sigma - g_3 \right\}^{-1/2} \, d\sigma \]
where
\[ g_2 = 3 A_2^2 - 4 A_1 A_2 \ , \ g_3 = 2 A_1 A_2 A_3 - A_2^3 - A_0 A_3^2 \]  
On substituting (11) and (12) into (17), (8) and (9) follow. From (16), it follows that
\[ s = \mathcal{P}(z) \]
with invariants \( g_2 \) and \( g_3 \). From the transformations given above,
\[ x = x_0 + \frac{1}{4} f'(x_0) \left\{ \mathcal{P}(z; g_2, g_3) - \frac{1}{24} f''(x_0) \right\}^{-1} \]  
\[ x = x_0 + \frac{1}{4} f'(x_0) \]  

So the lemma is now proved.

**Definition of Quasi-periodic function** \( \zeta(z) \): We introduce \( \zeta(z) \) through the equation

\[
\frac{d\zeta}{dz} = -\mathcal{P}(z)
\]  

(20)

coupled with the condition that

\[
\lim_{z \to 0} [\zeta(z) - z^{-1}] = 0
\]  

(21)

**Note:** Since the series for \( \mathcal{P}(z) - z^{-2} \) converges uniformly throughout any domain from which we exclude the neighborhoods of the points \( \Omega_{m,n} \) (i.e. not including \( m = n = 0 \) term), we get

\[
\zeta(z) - z^{-1} = -\int_0^z \{\mathcal{P}(z) - z^{-2}\} \, dz = -\sum_{m,n} \int_0^z \left\{(z - \Omega_{m,n})^{-2} - \Omega_{m,n}^{-2}\right\} \, dz
\]  

(22)

Hence

\[
\zeta(z) = \frac{1}{z} + \sum_{m,n} \left\{\frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2}\right\}
\]  

(23)

It is easily seen that the general term of this series is \( O(\Omega_{m,n}^{-3}) \) as \( |\Omega_{m,n}| \to \infty \) and hence the series in (23) defines an analytic function over whose \( z \)-plane except at simple poles, with residues 1, at all the points of the set \( \Omega_{m,n} \). It is easily shown that

\[
\zeta(-z) = -\zeta(z)
\]  

(24)

and hence it is an odd-function. We now show below the quasi-periodicity of \( \zeta(z) \): On integrating the relation

\[
\mathcal{P}(z + 2\omega_1) - \mathcal{P}(z) = 0
\]  

(25)

it follows that

\[
\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1
\]  

(26)

Evaluating this relation at \( \zeta = -\omega_1 \) and using the oddness property of \( \zeta \), it follows that

\[
\eta_1 = \zeta(\omega_1)
\]  

(27)

In like manner,

\[
\zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2
\]  

(28)

where

\[
\eta_2 = \zeta(\omega_2)
\]  

(29)
Lemma 4.4: For $\eta_1$ and $\eta_2$ defined as above,

$$\eta_1 \omega_2 - \eta_2 \omega_1 = \frac{1}{2} \pi i$$  \hspace{1cm} (30)

**Proof:** To obtain this relation, consider $\int_C \zeta(z) \, dz$, taken around the boundary of a cell. There is one pole of $\zeta(z)$ inside the cell with residue +1; hence $\int_C \zeta(z) \, dz = 2 \pi i$. However, the contour integral can be alternately expressed as:

$$2\pi i = \int_t^{t+2\omega_1} [\zeta(z)-\zeta(z+2\omega_2)] \, dz - \int_t^{t+2\omega_2} [\zeta(z)-\zeta(z+2\omega_1)] \, dz = -4\eta_2 \omega_1 + 4\eta_1 \omega_2$$  \hspace{1cm} (31)

Thus, the relation (30) follows.