1 Matched asymptotic expansion: preliminaries

The object of this note is to develop the Kaplun extension Theorem that is the key to existence of a matching region. This particular development of the subject is due to P. Lagerstrom.
Remark: We will now consider asymptotic expansion of a function in the form $u(x,\ \epsilon)$ where ϵ is a small parameter. Such functions appear naturally in the study of differential equations.

Definition 17.1:

$$\delta(\epsilon)\ =\ O_s(\eta(\epsilon)) \tag{1}$$

iff $\delta\ =\ O(\eta)$ and $\eta\ =\ O(\delta)$ as $\epsilon\ \to\ 0$. This means δ is *strictly* order η. We note that unlike the symbol O, O_s has a symmetrical property, i.e. if $\delta\ =\ O_s(\eta)$, then $\eta\ =\ O_s(\delta)$.

Definition 17.2:

$$ord\ \delta(\epsilon)\ =\ \{\eta(\epsilon)\mid \eta(\epsilon)\ =\ O_s(\delta(\epsilon))\ \}\quad \text{as}\ \ \epsilon\ \to\ 0 \tag{2}$$

Definition: M is an order class of functions of ϵ iff

$$\eta(\epsilon)\ \in\ M\ \ ,\ \ \delta(\epsilon)\ \ \in\ M\quad \text{implies}\ \ \eta\ =\ O_s(\delta)$$

Definition 17.3: Let $\{\zeta_j(\epsilon)\ \}_{j=0}^{\infty}$ be an asymptotic sequence, i.e. $\zeta_{j+1}\ <<\ \zeta_j$ for all j. We say that the sequence $a_j(x,\ \epsilon)$ is an asymptotic sequence of approximations to $u(x,\ \epsilon)$ in the closed x interval D, iff for all j,

$$\lim_{\epsilon\ \to\ 0}\ \frac{|u(x,\epsilon)-a_j(x,\epsilon)|}{\zeta_j(\epsilon)}\ =\ 0 \tag{3}$$

Remark: The end points of the interval in the definition 17.3 can be made to depend on ϵ. Secondly, the asymptotic sequence $\zeta_j(\epsilon)$ is not unique.

Remark: A common method for constructing $a_j(x,\ \epsilon)$ is to find a sequence $\{f_j(x,\ \epsilon)\}$ such that

$$a_j(x,\ \epsilon)\ =\ \sum_{k=0}^{j}\ f_k(x,\ \epsilon) \tag{4}$$

If (3) holds, then

$$u(x,\ \epsilon)\ \sim\ \sum_{k=0}^{\infty}\ f_k(x,\ \epsilon) \tag{5}$$

A special example of this is when

$$u(x,\ \epsilon)\ \sim\ \sum_{k=0}^{\infty}\ \beta_k(\epsilon)\ f_k(x) \tag{6}$$

where $\{\beta_k(\epsilon)\}$ forms an asymptotic sequence.

1.1 Domains of validity. Overlap and matching

Consider a special example

$$u(x,\ \epsilon)\ =\ e^{-x/\epsilon}\ +\ x\ +\ \epsilon \tag{7}$$

and the two possible approximations one gets to $u(x,\ \epsilon)$, depending on whether x or $\tilde{x}\ =\ \frac{x}{\epsilon}$ is fixed as $\epsilon\ \rightarrow\ 0$. In the first case,

$$u(x,\ \epsilon)\ \sim\ f(x)\ =\ x \tag{8}$$

while in the second case

$$u(x,\ \epsilon)\ \sim\ g(\tilde{x})\ =\ e^{-\tilde{x}} \tag{9}$$

To order unity, (8) and (9) are valid for $0\ <\ x_0\ \leq\ x\ \leq\ 1$ and $0\ \leq\ \tilde{x}\ \leq\ \tilde{x}_0$, respectively, where x_0 and $\tilde{x}_0$ are constants independent of ϵ. The second interval is $O(\epsilon)$ in width on the x-scale and hence shrinks to zero with ϵ. Closer inspection, however, shows that the estimates of validity are too conservative. First, on given intervals, the expansions are actually valid for any order $>>\ \epsilon$, i.e.

$$\lim_{\epsilon\rightarrow 0}\ \frac{u(x,\epsilon)-f(x)}{\zeta(\epsilon)}\ =\ 0 \tag{10}$$

and

$$\lim_{\epsilon\rightarrow 0}\ \frac{u(\epsilon\ \tilde{x},\epsilon)-g(\tilde{x})}{\zeta(\epsilon)}\ =\ 0 \tag{11}$$

for any $\zeta(\epsilon)\ >>\ \epsilon$. Secondly, it is possible to extend the interval on which $f(x)$ is valid to certain intervals whose left end point goes to zero with ϵ. This will be at the cost of having error increase, but the error will still be $<<\ 1$, provided the left end point does not go to zero too rapidly. For example, on the interval $\sqrt{\epsilon}\ \leq\ x\ \leq\ 1$, the maximum of $u(x,\ \epsilon)\ -\ f(x)$ is $exp(-1/\sqrt{\epsilon})\ +\ \epsilon$, which is still $O(\epsilon)$. However, on the interval $\epsilon\ \leq\ x\ \leq\ 1$, the maximum error is $1/e\ +\ \epsilon$, and $f(x)$ is not uniformly valid to $O(1)$ on this interval. Similarly $g(\tilde{x})$ is a uniformly valid approximation to $O(1)$ on $0\ \leq\tilde{x}\ \leq\ \eta(\epsilon)$, where $\eta(\epsilon)\ <<\ \frac{1}{\epsilon}$. In terms of x, this says that $g(\tilde{x})$ is uniformly valid for any interval whose right end point shrinks to zero. Therefore, consideration of uniform validity of approximations in an interval D, whose end points can depend on ϵ is important.

Definition 17.4 Two order class M and N satisfy the relation

$$M\ <\ N \tag{12}$$

iff for any $\delta(\epsilon)\ \in\ M$, $\eta(\epsilon)\ \in\ N$,

$$\delta(\epsilon)\ <<\ \eta(\epsilon)\quad \text{as}\ \ \epsilon\ \rightarrow\ 0 \tag{13}$$

Definition 17.5: We can define a closed interval D within the order classes of functions of ϵ to denote

$$D\ =\ \{Q\ |\ M\ \leq\ Q\ \leq\ N\} \tag{14}$$

Definition 17.6: An approximation $f(x,\ \epsilon)$ is said to be an approximation uniformly valid in D to $O(\zeta(\epsilon))$ if

$$\frac{|u(x,\epsilon)\ -\ f(x,\ \epsilon)|}{\zeta(\epsilon)}\ \rightarrow\ 0,\quad \text{uniformly in}\ \ D \tag{15}$$

This means that if $D = [M, N]$, and $\mu(\epsilon) \in M$ and $\nu(\epsilon) \in N$, then (15) is valid uniformly for x in $[\mu(\epsilon),\ \nu(\epsilon)]$. Clearly if (15) holds, we can replace $\zeta(\epsilon)$ in the denominator by any other $\zeta^*(\epsilon) >> \zeta(\epsilon)$.

Remark: The union of all such domains of validity is the *maximal* domain of validity. The maximal domain of validity of $g(\tilde{x})$ as an approximation to $u(x,\ \epsilon)$ to order unity (in the above example) is

$$D_g = \{\eta(\epsilon) \mid \eta(\epsilon) << 1\} \tag{17}$$

D_f, the maximal domain of validity of $f(x)$ to order unity, is harder to characterize, but certainly contains $\{\epsilon^a\}$, where $0 \le a < 1$ and all other functions that are o-equivalent to these.

Definition 17.7: The intersection of a domain of validity of f with a domain of validity of g is called an *overlap domain to order unity.* In such an overlap domain, both approximations are uniformly valid to order 1.

Remark: These definitions can be translated back into statements about x-domains of validity by picking a representative from D_f or D_g, say $\eta(\epsilon)\ x_\eta$, with x_η fixed. Clearly, picking one from each domain can cover the entire x-interval $0 \le x \le 1$ by a suitable pair of intervals with moving end points. In this sense we can use the notation

$$D_f\ U\ D_g = [0,\ 1] \tag{17}$$

Definition 17.8 Two valid approximations to $O(\zeta(\epsilon))$ of a function $u(x,\ \epsilon)$ with an overlap domain are said to *match* to $O(\zeta(\epsilon))$.

Lemma 17.1: If $f(x,\ \epsilon)$ and $g(x,\ \epsilon)$ are approximations to $u(x,\ \epsilon)$ with an overlap domain D to order $\zeta(\epsilon)$, and if η is in D, then for x_η, defined as $x = \eta(\epsilon)\ x_\eta$, fixed

$$lim_{\epsilon \to 0} \frac{f(x, \epsilon) - g(x,\ \epsilon)}{\zeta(\epsilon)} = 0 \tag{18}$$

Proof follows simply from the definition.

Remark: There is notation $\lim_\eta$ to the particular limiting process above, where x_η is held fixed with $\epsilon \to 0$. So in this notation, (18) can be written as

$$lim_\eta \frac{f(x, \epsilon) - g(x,\ \epsilon)}{\zeta(\epsilon)} = 0 \tag{19}$$

Theorem 17.1: (Kaplun's Extension Theorem Let M and N be two order classes with $M \le N$ and $f(x,\ \epsilon)$ an approximation to $u(x,\ \epsilon)$ valid to order $\zeta(\epsilon)$ in the order domain $[M,\ N]$. Then there exists order classes $M_e < M$ and $N_e > N$ such that $f(x,\ \epsilon)$ is an approximation to $u(x,\ \epsilon)$ valid to order $\zeta(\epsilon)$ in the extended order domain $[M_e,\ N_e]$.

Proof: We shall only prove the existence of M_e. The same reasoning will apply with obvious changes, to N_ϵ. Also, for simplicity of proof, we will be assume that M is the class of function of order unity

$$M = ord\ 1 = \{\eta \mid \eta = O_s(1)\} \tag{20}$$

There is no loss of generality, since any case may be reduced to this case by rescaling x with an arbitrary element of M.

Essential for the proof is that while M, as defined in (a), does not contain any function which tends to zero with ϵ, it contains a sequence of constant functions $a_n > 0$, which tend to zero with n, for instance $a_n = 1/n$. We define

$$w(x,\ \epsilon) \ = \ \frac{|u(x,\ \epsilon) - f(x,\ \epsilon)|}{\zeta(\epsilon)} \tag{21}$$

Consider the intervals $[a_n\ ,\ C]$, where C is an arbitrary constant $> a_1$. (Since we are only concerned with extending $[M,\ N]$ to the left, it is not necessary to have an element of N on the right end point. The proof is also valid when $M = N$.) For each a_n, there is by assumption an ϵ_n such that $w < a_n$ in the rectangle bounded by horizontal sides $\epsilon = 0$ and $\epsilon = \epsilon_n$ and the vertical sides $x = a_n$ and $x = C$ (See Fig. 1). One may assume $\epsilon_{n+1} < \epsilon_n$ and $\epsilon_n \to 0^+$ as $n \to \infty$.

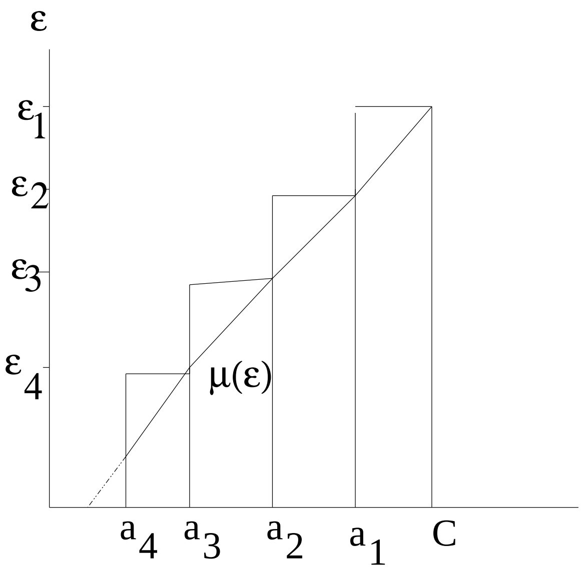

Figure 1: Extension of domain of validity

Now define

$$\mu(\epsilon_{n+1}) \ = \ a_n \tag{22}$$

and complete the definition of $\mu(\epsilon)$ as a piecewise linear function, as shown in Fig. 1. Clearly $w(x,\ \epsilon)$ tends to zero in the interval $[\mu(\epsilon),\ C]$, since given $\delta > 0$, we only need to find $a_n < \delta$. Then $w < a_n < \delta$ on any closed interval $[\mu(\epsilon),\ C]$ if $\epsilon \leq \epsilon_n$. Further

$$\mu(\epsilon) \ << \ 1, \quad \text{as} \ \ \epsilon \to 0 \tag{23}$$

It remains to show that w tends to zero uniformly in any interval $[\tilde{\mu}\ ,\ C]$, if $\tilde{\mu} = O_s(\mu)$. It is sufficient to choose

$$\tilde{\mu}(\epsilon) \ = \ \mu_N(\epsilon) \ = \ \frac{\mu(\epsilon)}{N} \tag{24}$$

for a positive integer N. To complete the proof, one replaces $\mu(\epsilon)$ in the proof above, for any fixed N, by

$$\mu_N(\epsilon_n) \;=\; \frac{a_(n-1)}{N} \tag{25}$$

Thus the order class M_e of the theorem is $M_e \;=\; ord\ \mu(\epsilon)$.

2 Illustration of asymptotic matching for a differential equation:

Example 1: Consider $u(x,\epsilon)$ in [0, 1] interval satisfying:

$$\epsilon\,\frac{d^2u}{dx^2} \;+\; \frac{du}{dx} \;-\; a \;-\; 2\,b\,x \quad , \quad u(0) \;=\; 0 \quad , \quad u(1) \;=\; 1 \tag{1}$$

where $0 \;<\; \epsilon \;<<\; 1$. The problem (1) has explicit solution:

$$u(x,\ \epsilon) \;=\; (1-a-b-2\epsilon\ b)\,\frac{(1-e^{-x/\epsilon})}{(1-e^{-1/\epsilon})} \;+\; a\ x \;+\; b\ x^2 \;-\; 2\ \epsilon\ b\ x$$

$$\;=\; \tilde{u}(x,\ \epsilon \;+\; O(e^{-1/\epsilon}) \tag{2}$$

where $\tilde{u}$ is obtained from u by neglecting transcendentally small term $e^{-1/\epsilon}$ in the denominator of the first term.

Solution by matched asymptotic expansions: If we assume that $u(x,\ \epsilon)$ has an asymptotic series of the form

$$u(x,\ \epsilon) \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\ u_j(x) \tag{3}$$

we find that u_0 obeys a first-order differential equation; hence the two boundary conditions at the two end points 0 and 1 would be an overspecification– they can only be satisfied accidentally. This suggests that even to the leading order, the term $\epsilon\,\frac{d^2u}{dx^2}$ must play a role. This means that the second derivative must at least be large in some region. If the linear homogeneous operator had been something like $\mathcal{L} \;=\; \left(\epsilon\,\frac{d^2}{dx^2} \;+\; 1\right)$, we would have rapid oscillations everywhere. But this cannot be the case in (1), since the roots of the characteristic for the homogeneous version of (1) are both real. We therefore assume that there is a layer of rapid change somewhere. In the limit $\epsilon \;\rightarrow\; 0^+$, this layer is expected to become a dicontinuity. If the limiting point of discontinuity occurs at $x \;=\; x_d$, we may formalize the concept of rapid change by introducing a scaled variable

$$\tilde{x} \;=\; (x-x_d)\ \epsilon^{-s} \quad \text{for} \;\; s \;>\; 0 \tag{4}$$

where ϵ^s is called a scaling parameter characterizing the thickness of the layer. In other examples, we may need to use more general dependence $\eta(\epsilon)$ for the thickness of the layer. In the layer, x changes little, and the essential behavior of the solution should be described by a function of $\tilde{x}$, whose derivatives are $O(1)$ on the $\tilde{x}$ scale. Now, we have to determine s and x_d. We shall justify the choice

$$s \;=\; 1 \quad , \quad x_d \;=\; 0 \quad \text{and hence} \quad \tilde{x} \;=\; x \,\epsilon^{-1} \tag{5}$$

An essential condition is that in some sense the layer solution (inner-solution, whose leading term is denoted by $g_0(\tilde{x})$) can be joined to the solution in x obtained by neglecting the second derivative (outer solution, whose leading order term is denoted by $f_0(x)$). This joining is known as matching. To find an equation for g_0, we introduce $\tilde{x}$ into (1) and let $\epsilon \;\rightarrow\; 0$, while keeping $\tilde{x}$ fixed. This is known as the inner-limit of the equation. The choice $s \;=\; 1$ is distinguished by the fact that it includes both terms on the left of (1) involving u, and the leading order equation is then given by

$$\frac{d^2\, g_0}{d\tilde{x}^2} \;+\; \frac{dg_0}{d\tilde{x}} \;=\; 0 \tag{6}$$

The solution to (6) is called the first term of the inner-expansion or solution. Its general solution grows exponentially as $x - x_d$ decreases and decays exponentially to a constant as $x - x_d$ increases. As may be anticipated, and verified later, the first fact precludes the possibility of matching. This is the reason for putting $x_d \;=\; 0$. The solution to (6) should be valid in a layer of thickness ϵ near $x \;=\; 0$. Thus, it should satisfy $g_0(0) \;=\; 0$, but does not have to satisfy the condition at $x \;=\; 1$, since that is outside the layer around $x \;=\; 0$ where the inner equation (6) is valid.

Except for the layer near $x \;=\; 0$, where strong variations occur, we hypothesize that in the rest of the domain $(0,\ 1)$, $\epsilon\, \frac{d^2 u}{dx^2}$ is indeed small. This means that we get the leading order *outer* equation (i.e. when $x \;\neq\; 0$ is held fixed and $\epsilon \;\rightarrow\; 0$) becomes:

$$\frac{d\, f}{dx} \;-\; a \;-\; 2\, b\, x \;=\; 0 \tag{7}$$

with the boundary condition

$$f_0(1) \;=\; 1 \tag{8}$$

The variable f_0 is called the leading term of the outer expansion. The variables $\tilde{x}$ and x are called the inner and outer variables respectively.

Since we now have only one boundary condition, we find from (7) and (8) that

$$f_0(x) \;=\; 1 \;-\; a \;-\; b \;+\; a\, x \;+\; b\, x^2 \tag{9}$$

The original trouble with putting $\epsilon \;=\; 0$ in (1) was that we obtained a first order differential equaiton with two boundary conditions. This has not been resolved by giving up the inner condition (i.e. condition $u(0) \;=\; 0$. However, for g_0, the opposite type of difficulty occurs. (6) is a second order equation but has only one boundary condition. The solution, within an unknown constant C_0 is given by

$$g_0(\tilde{x}) \;=\; C_0\, \left(1 - e^{-\tilde{x}}\right) \tag{10}$$

We now like to find C_0 by applying matching of the inner and outer expansions, based on the principle that if the true solution $u \;\sim\; g_0(\tilde{x})$ for some choice of C_0 for x in a domain $[0,\ \epsilon]$ and $u \;\sim\; f_0(x)$ for x in the region $[x_0\ ,\ 1]$ for $x_0 \;>\; 0$ (independent of ϵ), then from Kaplan's extension theorem, each of these domains can be extended so that there is some overlap domain $[\mu(\epsilon),\ \nu(\epsilon)]$ for $\epsilon \;<<\; \mu \;<\; \nu \;<<\; 1$, where both solutions are valid and must describe the same solution. In this domain, $f_0 \;\sim\; (1 -\; a \;- b)$, where as $g_0 \;\sim\; C_0$. Therefore matching requires that

$$C_0 \;=\; 1 \;-\; a \;-\; b \tag{11}$$