1 Higher order WKB

Recall in the last class, we used formal arguments to show how $n$ independent solutions to

$$L y \equiv e^n y^{(n)} - U(x) y(x) = 0$$

for $U(x) > 0$ has asymptotic behavior for small $\epsilon$:

$$y(x) \sim g_j [1 + o(1)] \quad \text{where} \quad g_j \equiv \exp \left[ \omega_j \int_{x_0}^x [U(s)]^{1/n} ds \right] [U(x)]^{-(n-1)/2n}$$

where

$$\omega_j^n = 1 \quad \text{for} \quad j = 1, 2, \ldots n$$

We now wish to rigorously justify this result. One way to prove this result is to determine a common equation satisfied by every $g_j$. That equation, as will turn out is close to (1) for small $\epsilon$. Then, using variation of parameter formula, we arrive at an integral equation representation for $y$ and then prove contraction.

1.1 Determination of common equation satisfied by $g_j$

For simplicity, we will limit ourself for $n = 3$, though in the process it will be clear that the results are generalizable for any $n$. In that case, it is convenient to take $\omega_j = e^{i 2j \pi / 3}$. It is to noted that $\Re \omega_{1,2} < 0$ and $\omega_3 = 1$. We will define $\beta = \frac{1}{\epsilon}$ and so $\beta >> 1$.

$$P(x) = \int_{x_0}^x [U(s)]^{1/n} ds$$

$$L(x) = [U(x)]^{(1-n)/(2n)}$$

Then, it is to be noted that

$$g_j(x) = L(x) \exp [\omega_j \beta P(x)]$$

It is convenient to define for $j = 1, 2, 3$:

$$m_{2,j} = \frac{\beta g_j}{\omega_j} = \omega_j P + \frac{L'}{\beta L} ; \quad m_{3,j} = \frac{g_j''}{\beta^2 g_j} = \left( \omega_j P + \frac{L'}{\beta L} \right)^2 + \beta^{-1} \left( \omega_j P' + \left[ \frac{L'}{\beta L} \right] \right)$$

and $r_j$ so that

$$L g_j = \beta^{-2} r_j g_j$$

Straight forward calculations (using maple) show that

$$r_j = \frac{7}{9} \omega_j \left( \frac{dU}{dx} (x) \right)^2 - \frac{2 \omega_j \frac{d^2 U}{dx^2} (x)}{3 \left( U(x) \right)^{2/3}}$$

$$\quad + \beta^{-1} \left[ \frac{28}{27} \left( \frac{dU}{dx} (x) \right)^3 + \frac{4}{3} \left( \frac{d^2 U}{dx^2} (x) \right) \frac{d^2 U}{dx^2} (x) - \frac{1}{3} \frac{d^3 U}{dx^3} (x) \right]$$

1
It is to be noted that if we define matrix $\mathcal{M}$ so that

$$
\mathcal{M} = \begin{bmatrix}
g_1 & g_2 & g_3 \\
\beta^{-1}g'_1 & \beta^{-1}g'_2 & \beta^{-1}g'_3 \\
\beta^{-2}g''_1 & \beta^{-2}g''_2 & \beta^{-2}g''_3
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix} \begin{bmatrix}
g_1 & 0 & 0 \\
0 & g_2 & 0 \\
0 & 0 & g_3
\end{bmatrix}
$$

(10)

Then, it is convenient to define matrix $Q_1$ so that

$$
\beta^{-1}Q_1 \equiv (\mathcal{M}' - \beta Q_0 \mathcal{M}) \mathcal{M}^{-1}
$$

where

$$
Q_0 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
U & 0 & 0
\end{bmatrix}
$$

(12)

Using (8), (11) and (12), it is seen that

$$
\beta^{-1}Q_1 \mathcal{M} = \mathcal{M}' - \beta Q_0 \mathcal{M} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\beta^{-1}r_1 & \beta^{-1}r_2 & \beta^{-1}r_3
\end{bmatrix} \begin{bmatrix}
g_1 & 0 & 0 \\
0 & g_2 & 0 \\
0 & 0 & g_3
\end{bmatrix}
$$

(13)

From (10) and (13), it follows that

$$
Q_1 = \begin{bmatrix}
1 & 1 & 1 \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix} \begin{bmatrix}
g_1 & 0 & 0 \\
0 & g_2 & 0 \\
0 & 0 & g_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
r_1 & r_2 & r_3
\end{bmatrix} \begin{bmatrix}
g_1 & 0 & 0 \\
0 & g_2 & 0 \\
0 & 0 & g_3
\end{bmatrix}
$$

(14)

Thus,

$$
Q_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
b_{31} & b_{32} & b_{33}
\end{bmatrix}
$$

(15)

where,

$$
b_{3,2} = [(r_1 - r_2)(m_{32} - m_{33}) - (r_2 - r_3)(m_{31} - m_{32})]/\Delta,
$$

(16)

$$
b_{3,3} = -[(r_1 - r_2)(m_{22} - m_{21}) - (r_2 - r_3)(m_{21} - m_{22})]/\Delta,
$$

(17)

$$
b_{3,1} = r_3 - b_{3,2}m_{2,3} - b_{3,3}m_{3,3},
$$

(18)

with $\Delta = (m_{21} - m_{22})(m_{32} - m_{33}) - (m_{22} - m_{23})(m_{31} - m_{32})$ we have Thus, from (11) it follows that the matrix $\mathcal{M}$ satisfies the differential equation

$$
\mathcal{M}' - (\beta Q_0 + \beta^{-1}Q_1) \mathcal{M} = 0
$$

(19)

In particular by looking at the third row of this equation, it is seen that for any $j$, $g_j$ satisfies the same third order homogeneous linear differential equation:

$$
\mathcal{L}_w g_j := \beta^{-3}g'''_j - \beta^{-1}b_{3,3}g''_j - \beta^{-3}b_{3,2}g'_j - [U(x) + \beta^{-2}b_{31}]g_j = 0
$$

(20)

Notice that $\mathcal{L} - \mathcal{L}_w$ is indeed small and scales asmost like $\beta^{-2}$. This can be used to prove a contraction mapping and thereby justify the WKB approximation.
2 Formal Boundary Layer Analysis

Example: Linear second order Equations with variable coefficients:

\[ \epsilon u'' + a(x) u' + b(x) u = 0 \]  \hspace{1cm} (21)
\[ u(x_0) = A, \quad u(x_1) = B \]  \hspace{1cm} (22)

Here \( x_0 < x_1 \), and prime denotes \( \frac{d}{dx} \). The functions \( a(x) \) and \( b(x) \) are as smooth as necessary to carry out the steps below. Without loss of generality, we take \( x_0 = 0 \) and \( x_1 = 1 \).

**Comment:** We note that by applying transformation

\[ u(x) = \exp \left[ \frac{-1}{2\epsilon} \int_0^x a(t) \, dt \right] w(x) \]  \hspace{1cm} (18)

the (21) becomes

\[ \epsilon^2 w'' + \left( -\frac{a^2}{4} + \epsilon b - \frac{1}{2} \epsilon a' \right) w = 0 \]  \hspace{1cm} (23)

In this form, the WKBJ method is immediately applicable and one can obtain approximate solutions as \( \epsilon \to 0 \). However, we like to directly find approximate solution to (21), though in a formal manner, using the so-called boundary layer method.

**Solution to (21)-(22) for \( a(x) > 0 \)**

We assume there is an an outer region where for fixed \( x \), as \( \epsilon \to 0 \), the solution to (21) has the asymptotic expansion:

\[ u \sim \sum_{j=0}^{\infty} \epsilon^j u_j(x) \]  \hspace{1cm} (24)

The equations for \( u_0 \) is

\[ a(x)u'_0 + b(x) u_0 = 0 \]  \hspace{1cm} (25)

The general solution to this is of the form

\[ u_0(x) = C \exp \left( \int_{x_1}^{x} \frac{b(t)}{a(t)} \, dt \right) \]  \hspace{1cm} (26)

for some constant \( C \) that is undetermined at this stage. There are now two boundary conditions in (22). Which one needs to be satisfied by \( u_0 \) depends on the location of the boundary layer (referred to as a thin region in \( x \) where solution changes dramatically). We do not know ahead of time where this layer is, if at all there is a boundary layer. Suppose, it is centered around some point \( x_d \) in the interval \( [x_0, x_1] \). We rescale variable:

\[ \tilde{x} = \frac{(x - x_d)}{\eta(\epsilon)} \]  \hspace{1cm} (27)

Then, (21) becomes

\[ \frac{\epsilon}{\eta^2} \frac{d^2 u}{d\tilde{x}^2} + \frac{1}{\eta} \left[ a(x_d) + a'(x_d) \eta \tilde{x} + O(\eta^2) \right] \frac{du}{d\tilde{x}} + [b(x_d) + O(\eta)] u = 0 \]  \hspace{1cm} (28)
Now two distinguished limits are

\[ \eta(\epsilon) = 1 \quad \text{and} \quad \eta(\epsilon) = \epsilon \]  

(29)

The first reproduces the original equation (21). The second choice, gives rise to the inner-equation:

\[
\frac{d^2 u}{d\tilde{x}^2} + \left[ a(x_d) + a'(x_d) \epsilon \tilde{x} + O(\epsilon^2) \right] \frac{du}{d\tilde{x}} + \epsilon \left[ b(x_d) + O(\epsilon) \right] u = 0
\]  

(30)

Now, if we assume an inner expansion of the type:

\[ u(x, \epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j U_j(\tilde{x}), \]  

(31)

then \( U_0(\tilde{x}) \) satisfies

\[ U''_0 + a(x_d) U'_0 = 0 \]  

(32)

A general solution to (32) is of the form

\[ U_0(\tilde{x}) = C_1 - C_0 e^{-\tilde{x} a(x_d)} \]  

(33)

It is to be noted that for \( C_1 \neq 0 \), for \( \tilde{x} = (x - x_d)/\epsilon \rightarrow -\infty \), the solution (33) is dominated by an exponentially growing term \( e^{-a(x_d)} (x-x_d)/\epsilon \) which is not present in the outer solution \( u_0(x) \); hence there is no possibility of an overlap domain to the left of \( x = x_d \). Thus, consistency eliminates choice of \( x_d \neq 0 \). From now on, we continue with the choice \( x_d = 0 \). Thus, if there should be a boundary layer, it must be on the left end point 0. The leading order outer solution (26) must therefore satisfy the boundary condition on the right, i.e. \( u_0(1) = B \). This is possible when \( C = B \) in the expression (26). From the leading order inner solution (33), now rewritten with \( x_d = 0 \), it follows that boundary condition \( u(0) = A \) translates into \( U_0(0) = A \) and therefore constant \( C_1 \) has to be chosen so that

\[ U_0(\tilde{x}) = A + C_0 \left[ 1 - e^{-a(0)} \tilde{x} \right] \]

(34)

Leading order outer solution (26) and inner solution (34) match in an overlap domain \( \epsilon << x << 1 \), provided

\[ C_0 + A = B \exp \left[ \int_0^1 \frac{b(t)}{a(t)} \, dt \right] \]  

(35)

Next time, we will consider higher order approximations.

3 Higher Order Approximation and Matching

Recall the ansatz for the outer

\[ u \sim \sum_{j=0}^{\infty} \epsilon^j u_j(x) \]  

(36)

Plugging into the equation, it follows that for \( j \geq 1 \),

\[ a(x)u'_j + b(x)u_j = -u_{j-1} \]

(37)
We determined last time that boundary layer can only occur at \( x = 0 \), so the boundary conditions at \( x = 1 \) have to be imposed on the outer-solution; i.e.

\[
    u_0(1) = B, \quad u_j(1) = 0 \quad \text{for } j \geq 1
\]  

(38)

As found earlier,

\[
    u_0(x) = B \exp \left[ \int_x^1 \frac{b(t)}{a(t)} \, dt \right]
\]  

(39)

Plugging this into the right hand side of (37) for \( j = 1 \), and using the method of integrating factors, we obtain

\[
    u_1(x) = \left\{ \exp \left[ \int_x^1 \frac{b(t)}{a(t)} \, dt \right] \right\} \left\{ \int_x^1 \left[ \frac{u_0''(t)}{a(t)} \exp \left( - \int_t^1 \frac{b(s)}{a(s)} \, ds \right) \right] \, dt \right\}
\]  

(40)

As determined last time, the inner variable choice is

\[
    \tilde{x} = \frac{x}{\epsilon}
\]  

(41)

and the inner equation becomes

\[
    \frac{d^2 u}{d\tilde{x}^2} + a(\epsilon \tilde{x}) \frac{d u}{d\tilde{x}} = - b(\epsilon \tilde{x}) u
\]  

(42)

If we assume that \( a \) and \( b \) to be smooth near the origin, then the inner expansion takes the form:

\[
    u(x, \epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j U_j(\tilde{x})
\]  

(43)

More generally, if \( a(x) \) and \( b(x) \) are functions in \( C^0[0,1] \) space, then the asymptotic expansion (43) remains valid up to \( j = n \) term. Making of the local expansion of \( a \) and \( b \), we obtain as before:

\[
    U_0'' + a_0 U_0' = 0 \quad \text{with} \quad U_0(0) = A,
\]  

(44)

\[
    U_1'' + a_0 U_1' = - a_1 \tilde{x} \frac{d U_0}{d \tilde{x}} - b_0 U_0, \quad U_1(0) = 0,
\]  

(45)

where \( a_0 = a(0), \ a_1 = a'(0), \ b_0 = b(0) \). A general solution to (44) satisfying given boundary condition is of the form

\[
    U_0(\tilde{x}) = A + C_0 \left( 1 - e^{-a_0 \tilde{x}} \right)
\]  

(46)

In order for matching between \( U_0 \) and \( u_0 \) to occur, we must have

\[
    C_0 = k - A \quad \text{where} \quad k = B \exp \left[ \int_0^1 \frac{b(t)}{a(t)} \, dt \right]
\]  

(47)

The solution to \( U_1(x) \) can be obtained by solving (45), with given right hand side and we get

\[
    U_1(\tilde{x}) = - \frac{1}{2} a_1 (A-k) \tilde{x}^2 e^{-a_0 \tilde{x}} - \frac{1}{a_0} (A-k) (a_1 - b_0) \tilde{x} e^{-a_0 \tilde{x}} + C_1 (1 - e^{-a_0 \tilde{x}}) - \frac{b_0 k}{a_0} \tilde{x}
\]  

(48)
Matching \( u_0 + \epsilon u_1 \) with \( U_0 + \epsilon U_1 \) gives

\[ C_1 = u_1(0) \]  \hspace{1cm} (49)

Other terms cancel out exactly. The composite (uniformly valid) expansion to order \( \epsilon \) is given by adding \( u_0 + \epsilon u_1 \) to \( U_0 + \epsilon U_1 \) and subtracting out the terms that match in the overlap domain. This is given by

\[ u(x, \epsilon) \sim u_0(x) + \epsilon u_1(x) + (A-k) e^{-a_0 \tilde{x}} + \epsilon \left[ -\frac{1}{2} a_1 (A-k) \tilde{x}^2 - \frac{1}{a_0} (A-k) \tilde{x} - u_1(0) \right] e^{-a_0 \tilde{x}} \]  \hspace{1cm} (50)

**Case B:** \( a(0) = 0, \ a(x) > 0 \text{ for } x \in (0, 1) \)

Note that the arguments before can made again to show that there cannot be any boundary layer except possibly at \( x = 0 \). Thus, the outer solutions worked out above for \( u_0 \) and \( u_1 \) remain valid without any modifications. The inner equation near \( x = 0 \), however, is different both in the size of the boundary layers and the leading order equations that one gets. There are many possibilities, depending on the nature of zero of \( a(x) \) at \( x = 0 \). We illustrate the possibilities in terms of a special case of (21):

\[ \epsilon u'' + x^k u' + b u = 0 \hspace{0.5cm} , \hspace{0.5cm} b = \text{constant}, \hspace{0.5cm} k > 0 \]  \hspace{1cm} (51)

\[ u(0) = A, \ hspace{0.5cm} u(1) = B \]  \hspace{1cm} (52)

Clearly since only nature of inner solutions to (51) is closely related to that of the inner solutions to (21) if case B applies and \( a(x) \sim x^k \) as \( x \to 0^+ \). Thus the discussion here for (51) has general consequences for (21) as well, when \( a(x) \) is zero at one of the end points and nonzero elsewhere in the closed interval \([0, 1]\). If introduce an inner variable

\[ \tilde{x} = \frac{x}{\epsilon^l} \]  \hspace{1cm} (53)

into (50), then it becomes

\[ \frac{d^2 u}{d\tilde{x}^2} + \epsilon^{l(1+k)-1} \tilde{x}^k \frac{d u}{d\tilde{x}} + \epsilon^{2l-1} b u = 0 \]  \hspace{1cm} (54)

Different cases are possible, depending on the value of \( k \).

If \( 0 < k < 1 \) (we call this case B1), we choose

\[ l = \frac{1}{1+k} \]  \hspace{1cm} (55)

in (53). The first two terms in (54) are then order unity and the third term is \( O(\epsilon^{(1-k)/(1+k)}) = o(1) \).

For \( k > 1 \) (call it case B2), the choice (55) is inappropriate as this makes the third term in (54) unbalanced, which is not possible. So, we choose instead

\[ l = \frac{1}{2} \text{ for } k > 1 \]  \hspace{1cm} (56)

in (53). Then the first and third term in (54) are order unity, while the second term is \( o(1) \).
Finally, for \( k = 1 \) (called case B3), we choose \( l = \frac{1}{2} \) again and in that case all three terms in (54) are \( O(1) \) and need to be included.

Summarizing: Three distinct cases are B1: \( 0 < k < 1 \), B2: \( k \geq 1 \) and B3: \( k = 1 \). We now study case B1 in detail.

**Inner and outer solutions for case B1:**

From the equation for \( u_0 \) and \( u_1 \), we get

\[
\frac{d^2 u_0}{d\tilde{x}^2} + \tilde{x}^k \frac{du_0}{d\tilde{x}} + \zeta(\epsilon) \ b \ u = 0 \tag{61}
\]

where

\[
\tilde{x} = x \eta^{-1} = x \ e^{-1/(1+k)} , \quad \zeta(\epsilon) = e^{(1-k)/(1+k)} \tag{62}
\]

This suggests an expansion of the form

\[
u \sim \sum_{j=0}^{\infty} \zeta^j U_j(\tilde{x}) \tag{63}
\]

Note that the coefficients \( \zeta^j \) are integral powers of the scaling factor \( \eta \) or of \( \epsilon \) only for exceptional values of \( j \) and \( k \). The \( U_j(\tilde{x}) \) satisfy

\[
\frac{d^2 U_0}{d\tilde{x}^2} + \tilde{x}^k \frac{dU_0}{d\tilde{x}} = 0 \ , \quad U_0(0) = A \ , \tag{64}
\]

and for \( j > 0 \),

\[
\frac{d^2 U_j}{d\tilde{x}^2} + \tilde{x}^k \frac{dU_j}{d\tilde{x}} = -b \ U_{j-1} \ ; \quad g_j(0) = 0 \tag{65}
\]

The first equation is solved by

\[
U_0(\tilde{x}) = C_0 \ G(\tilde{x}) + A \ , \quad G(\tilde{x}) = \int_{0}^{\tilde{x}} e^{x} \exp \left( -t^{1+k}/(1+k) \right) \ dt \tag{66}
\]
where $C_0$ is obtained by matching $U_0$ with $u_0$. By breaking up the integral for $G$ into two parts: 
\[
\int_0^s = \int_0^\infty - \int_s^\infty,
\]
and using integration by parts on the second integral, it is clear that the latter integral is exponentially small in $\tilde{x}$. Thus, it is clear that the matching condition becomes:
\[
\lim_{\tilde{x} \to \infty} U_0(\tilde{x}) = C_0 (1 + k)^{-k/(1+k)} \Gamma(1/(1+k)) + A = \lim_{x \to 0} u_0(x) = B e^{b/(1-k)} \quad (67)
\]
This determines $C_0$. For $j > 0$, we find $U_j(\tilde{x})$ by the recursive formula:
\[
U_j(\tilde{x}) = \int_0^\tilde{x} \exp (-s^{1+k}/(1+k)) \left\{ \int_0^s [-bU_{j-1}(t) \exp (t^{1+k}/(1+k)) \ dt] \right\} ds + C_j G(\tilde{x}) \quad (68)
\]
In order matching of the inner and outer variables, it becomes necessary to study the asymptotics of (68) for large $\tilde{x}$. This is a bit tedious. We first study the behavior of the inner-integral in (68) by breaking up the integral and integrating one of them by parts:
\[
\int_0^1 U_{j-1}(t) \exp (t^{1+k}/(1+k)) \ dt + \frac{U_{j-1}(s)}{s^k} \exp \left[ \frac{s^{1+k}}{1+k} \right] - \frac{U_{j-1}(1)}{s^k} \exp \left[ \frac{1}{1+k} \right] \\
- \int_1^s U'_{j-1}(t) \exp (t^{1+k}/(1+k)) \ dt + \int_1^s k t^{-k} U_{j-1}(t) \exp (t^{1+k}/(1+k)) \ dt
\]
With this decomposition, it is clear that as $\tilde{x} \to +\infty$
\[
U_1(\tilde{x}) \sim -b U_0(\infty) \frac{\tilde{x}^{1-k}}{1-k} + \tilde{C}_1 + O(\tilde{x}^{-k}) , \quad (70)
\]
where
\[
\tilde{C}_1 = C_1 G(\infty) - b \int_0^\infty ds e^{-s^{1+k}/(1+k)} \left\{ \tilde{C} + s^{-k} \left[ U_0(s) - U_0(\infty) \right] \exp \left[ s^{1+k}/(1+k) \right] \\
+ \int_1^s \exp \left[ t^{1+k}/(1+k) \right] \left[ -U_0'(t) t^{-k} + k t^{-k-1} \right] dt \right\} \quad (71)
\]
where
\[
\tilde{C} = \int_0^1 U_0(t) \exp \left[ t^{1+k}/(1+k) \right] dt - U_0(1) \exp [1/(1+k)] \quad (72)
\]
Note that the term when $U_0(\tilde{x}) + \zeta U_1(\tilde{x})$ is rewritten in outer variables $x$, we obtain
\[
U_0(\infty) - b U_0(\infty) \frac{x^{1-k}}{1-k} + \zeta(\epsilon) \tilde{C}_1 + O(\epsilon x^{-2k}) \quad (73)
\]
In the outer asymptotic expansion $u_0(x) + \epsilon u_1(x) + ..$, it is clear from (59) and (60) that the terms that can match with (73) as as $x \to 0$ are
\[
B \exp \left[ b/(1-k) \right] \left( 1 - \frac{b}{1-k} x^{1-k} \right) + O(x^{2-2k}, \epsilon x^{-2k}) \quad (74)
\]
There are no terms that corresponds to a constant times $\zeta(\epsilon)$. This means that matching requires that
\[
\tilde{C}_1 = 0 \quad (75)
\]
This translates into a condition on $C_1$, since $C_1$ and $\tilde{C}_1$ are related through (71).