Solution to Homework Set 1, Math 805

1. Assume $f$ is entire, with $|f(z)| \leq C_1 e^{K|z|}$ for $z \in \mathbb{C}$ and $|f(z)| \leq C e^{-|z|}$ for in a sector with opening angle exceeding $\pi$. Show that $f(z) = 0$ for all $z$.

**Solution:**

Let $S$ denote the sector with opening angle exceeding $\pi$. Clearly this implies that there is a sector $S_1$ with opening angle less than $\pi$ so that $\partial S_1 \in S$ and $\partial S \in S_1$. Therefore, $|f(z)| \leq C$ on $\partial S_1$. From Phragmen Lindeloff principle, $|f| \leq C$ in $S_1$. Since this is true on $S$, it follows $f$ is bounded everywhere. Since it is entire, it can only be a constant. From the exponentially decaying estimate in $S$, that constant must be zero.

![Figure 1: Sectors $S$ and $S_1$ in problem 1.](image)

2. Assume $f$ is analytic for $|z| > z_0$ in a sector $S$ with opening angle more than $\pi$ and $|f(z)| \leq C e^{-a|z|}$ $(a > 0)$ in $S$. Show that $f$ is identically zero.

**Solution:** We may choose $\alpha > z_0$ and some $\delta > 0$ so that the shifted sector (see Fig 2)

$$S^\alpha_\delta = \{\text{arg}(z - \alpha) \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \}$$

is entirely contained in $S \cap \{|z| > z_0\}$. Define $Z = z - \alpha$, and $g(Z) = f(z)$. Consider

$$F(p) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{pz} f(z) dz = \frac{e^{\alpha p}}{2\pi i} \int_{-i\infty}^{i\infty} e^{pZ} g(Z) dZ = e^{\alpha p} G(p)$$

Since exponential decay of $f$ implies in particular $|g(Z)| \leq \frac{C}{1+|Z|^2}$ for $Z \in S$, it follows from Theorem in text that $G(p)$ (and therefore $F(p)$) is analytic in sector $S^\delta_\alpha$ with opening angle $\delta$. Further, if $|p| < \frac{\alpha}{2}$, it follows that

$$|e^{pz} f(z)| \leq e^{-a|z|/2}$$
on a right-half semi-circle $C_R$ that closes the contour $\int_{\alpha-iR}^{\alpha+iR}$ on the right. From calculus of residues, for $|p| < \frac{a}{2}$,

$$F(p) = \frac{1}{2\pi i} \int_{\alpha-iR}^{\alpha+iR} e^{pz} f(z)dz = 0$$

Therefore, from analytic continuation, $F(p) = 0$ for any $p$ including on $\mathbb{R}^+$. On Laplace transform $f(z) = 0$ for $z \in S^\delta_{\alpha}$ and by analytic continuation for any $z \in \mathbb{C}$.

3. Exercise 3.8, Page 26 Text: Consider the function

$$F(z) = e^{-z^2} \int_0^z s^{-2}e^{-s^2} ds$$

Show that $F$ has an isolated singularity at $z = 0$. Using Remark 1.32, show that $F$ is unbounded in some direction as $z \to 0$.

**Solution:** First, we show $F$ is analytic at $z \neq 0$. Easiest to change variables $x = z^{-2}$ and then note

$$F(1/x) = e^{x^2} \int_{+\infty}^{x} e^{-t^2} dt$$

It is clear that $\int_{+\infty}^{x} e^{-t^2} dt$ is analytic function of $x$ for any finite $x$ since $e^{-x^2}$ is. Therefore, from product rule, so is $F(1/x)$, implying $F$ is analytic and single valued in $z$ for $z \neq 0$. If $F$ were analytic at $z = 0$, it must have a convergent Taylor expansion at $z = 0$, which is also its asymptotic expansion at $z = 0$ (from Remark 1.32). However, we noted from expression (3.7) for the asymptotic expansion as $z \to 0^+$ that it is divergent. Therefore, $z = 0$ must be a isolated singular point of $F$. If $F$ had a finite limit in every direction, then its Laurent series at $z = 0$ will not have any singular parts; it would be a removable singularity, i.e. $F$ is analytic at $z = 0$. This is not the case.
4. Exercise 3.9, Page 26 Text: Given $x$, find $m = m(x)$ so that the accuracy in approximating $E(x) = e^{x^2} \int_x^\infty e^{-s^2} ds$ by the truncated series

$$E_m(x) \equiv \sum_{k=0}^m \frac{(-1)^k \Gamma \left( k + \frac{1}{2} \right)}{2 \sqrt{\pi} x^{2k+1}}$$

is the highest. Show that $m$ is approximately the one that minimizes the $m$-term of the series for given $x$ (i.e. the $m$-th term is the least term). For $x = 10$, the relative error in calculating $E$ in this way is $5.3 \times 10^{-42}\%$.

**Solution:** Note that the $m$-term is minimum for given $x$ (appropriately large) when

$$\frac{\Gamma(m + 1/2)}{x^{2m+1}}$$

is a minimum in $m$ (2)

We note from integration by parts that the error is

$$\epsilon_m = E - E_m = \frac{(-1)^m e^{x^2} \Gamma(m + 1/2)}{\sqrt{\pi}} \int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds$$

Notice on integration by parts the asymptotic behavior for large $x$ of

$$\int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds \sim \frac{e^{-x^2}}{2x^{2m+1}}$$

Therefore

$$\epsilon_m \approx \frac{(-1)^m \Gamma(m + 1/2)}{x^{2m+1} \sqrt{\pi}}$$

which is minimized when condition (2) is met.

To determine where minimum occurs, it is simple to take the ratio test of two neighboring terms of the asymptotic expansion and note when the absolute value of this ratio changes from being smaller than 1 to greater than 1 as we change $m \to m + 1$. Note the absolute value of the ratio is

$$\frac{\Gamma(m + 1/2) x^{2m-1}}{\Gamma(m - 1/2) x^{2m+1}} = \frac{m - 1/2}{x^2}$$

So $m = \left[ x^2 + \frac{1}{2} \right]$ will minimize the $m$-th term of the asymptotic expansion. Working this out for $x = 10$ gives $m = 100$. So,

$$|\epsilon_m| \approx \frac{\Gamma(100.5)}{\sqrt{\pi} 10^{201}} \approx 5.25 \times 10^{-45}$$