Solution to Homework Set 2, Math 805

1. (Exercise 1, Page 33, Text). How can this method (Borel-Ritt Lemma) be modified to give a function analytic in a sector of opening angle $2\pi n$ for an arbitrary fixed $n$ which is asymptotic to $\tilde{f}$.

**Solution**: We define $y$ as the independent variable. We seek analytic function $f$ in a sector $S$ of width $2\pi n$ so that as $y \to \infty$ in $S$,

$$f(y) \sim \tilde{f}(y) = \sum_{j=0}^{\infty} \frac{a_j}{y^{j+1}}$$

Without loss of generality, assume $S$ is centered about the positive real axis.

Define $x = y^{1/(2n)}$. The problem is then equivalent to finding analytic function $g$ in $\mathbb{H}$ with the property that $x \to \infty$ along a ray in $\mathbb{H}$,

$$g(x) \sim \tilde{f}(x^{2n}) = \sum_{j=0}^{\infty} \frac{a_j}{x^{2(j+1)n}}$$

We define

$$c_k = a_j \text{ for } k = 2n(j + 1) - 1 \text{ and } c_k = 0 \text{ otherwise}$$

It follows

$$\tilde{g}(x) \equiv \tilde{f}(x^{2n}) = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$$

We may now use

$$\tilde{G}(p) = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k$$

and construct an $L^1$ function $G(p)$ on $(0, p_0)$ in accordance to Proposition 3.40 in the text so that

$$G(p) \to \tilde{G}(p) \text{ as } p \to 0^+$$

Then, $g$ defined by

$$g(x) = \int_{0}^{p_0} G(p) e^{-px} dp$$

will be the entire function in $x$ with the desired asymptotic property as $x \to \infty$ in $\mathbb{H}$. Hence $f(y) = g(y^{1/(2n)})$ has the desired property that $f(y) \to \tilde{f}(y)$ as $y \to \infty$ in $S$.

2. (Exercise 3.68, page 39 Text) Show that if $f$ is analytic in a neighborhood of $[a, b]$ but not entire, both series

$$\frac{e^{ixb}}{ix} \sum_{n=0}^{\infty} \frac{(-1)^n f^{(n)}(b)}{(ix)^n}, \quad \frac{e^{ixa}}{ix} \sum_{n=0}^{\infty} \frac{(-1)^n f^{(n)}(a)}{(ix)^n}$$

have zero radius of convergence.
Solution: Suppose the series in equation (0) involving \( a \) had a nonzero radius of convergence. Then there exists finite \( \rho \) such that
\[
|f^{(n)}(a)\rho^{-n}| \leq 1
\]
for \( n \geq N \) large enough. Then, the series for \( f \): \( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \) has infinite radius of convergence since
\[
\sum_{n=0}^{\infty} \frac{|f^{(n)}(a)|}{n!}|z-a|^n \leq \sum_{n=0}^{\infty} \frac{\rho^n}{n!}|z-a|^n
\]
the latter being convergent for any \( z \in \mathbb{C} \) and \( f \) has to be entire contradicting our assumption. Clearly the same argument is valid when \( a \) is replaced by \( b \).

3. Determine the full asymptotic expansion as \( x \to \pm \infty \) of \( I(x) = \int_0^1 e^{xt^3} dt \)
Solution First, consider \( x \to +\infty \). Since \( t^3 \) is monotonically increasing, the full asymptotic series is determined from the a neighborhood of \( t = 1 \). We use a change of variable
\[
\tau = 1 - t^3 \quad \text{and note} \quad t = (1 - \tau)^{1/3}
\]
Therefore,
\[
I(x) = e^x \int_0^1 e^{-xt^3} dt = -\frac{e^x}{3} \int_0^1 e^{-x\tau(1-\tau)^{-2/3}} d\tau
\]
Using Watson’s Lemma
\[
I(x) \sim e^x \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}
\]
where
\[
c_k = -\frac{k!}{3} \left( -\frac{2}{3} \right)^k
\]
Now consider \( x = -y \to -\infty \). We have with \( \tau = t^3 \) and Watson’s Lemma:
\[
I = \int_0^1 e^{-yt^3} dt = \int_0^1 e^{-yt^3} dt d\tau = \frac{1}{3} \int_0^1 e^{-y\tau\tau^{-2/3}} d\tau \approx \frac{1}{3} \frac{\Gamma(1/3)}{y^{1/3}}
\]
Note that the entire asymptotic series is just one term in this case.

4. Employ Stationary phase method to determine first three terms of the asymptotic expansion of
\[
I(x) = \int_0^1 e^{ix(t-t^2/2)} dt \equiv \int_0^1 e^{ixh(t)} dt
\]
Solution:
We note \( \frac{dh}{dt} = 1 - t = 0 \) at the end point \( t = 1 \). Hence \( t = 1 \) is a stationary phase point. Hence, we decompose the integral \( I(x) \):
\[
I(x) = \int_0^1 e^{ixh(t)} dt + \int_1^1 e^{ixh(t)} dt \equiv I_1(x) + I_2(x), \tag{3}
\]
with the understanding that the eventual result cannot depend on artificial parameter \( \delta \), which is chosen independent of \( x \).

With choice of variable \( v = (1 - t) \), we note that

\[
I_2(x) = e^{ix/2} I_3(x),
\]

where

\[
I_3 = \int_0^{1-\delta} e^{-i x v^2} dv = \frac{1}{2\sqrt{x}} \int_0^{(1-\delta)x} e^{-iu} du \sim \frac{1}{2\sqrt{x}} \int_0^{\infty} e^{-iu} du + \text{terms depending on } \delta
\]

Using contour integration (closing the contour through a \( \pi/2 \) clockwise rotation and using Jordan’s Lemma), it follows that

\[
I_3 \sim \frac{e^{-i\pi/4} \Gamma(1/2)}{2\sqrt{x}} + \text{terms depending on } \delta
\]

The contribution from \( I_1 \) on integration by parts twice gives:

\[
I_1(x) \sim \int_0^{\delta} e^{ixh(t)} dt = \left( \frac{e^{ixh(t)}}{ixh'(t)} + \frac{e^{ixh(t)}}{x^2(h'(t))} \frac{d}{dt} h'(t) \right) \bigg|_0^\delta = -\frac{1}{ix} \frac{1}{x^2} + O(x^{-3}) + \text{terms depending on } \delta
\]

Therefore, since \( \delta \) dependent terms must cancel out, it follows that the first three terms of the asymptotic expansion is:

\[
I(x) = \frac{e^{ix/2-i\pi/4} \Gamma(1/2)}{2\sqrt{x}} - \frac{1}{ix} \frac{1}{x^2} + O(x^{-3})
\]

5. (Exercise 2, Page 33, Txt). Assume \( F \) is bounded on \([0, 1]\) and has an asymptotic expansion

\[
F(t) \sim \sum_{k=0}^{\infty} c_k t^k
\]

as \( t \to 0^+ \). Let \( f(x) = \int_0^1 e^{-xp} F(p) dp \).

(a.) Find necessary and sufficient conditions on \( F \) such that \( \tilde{f} \), the asymptotic power series of \( f \) for large positive \( x \), is a convergent series for \( |x| > R > 0 \).

(b.) Assume that \( \tilde{f} \) converges to \( f \). Show that \( f = 0 \).

(c.) Show that in case (a.), if \( F \) is analytic in a neighborhood of \([0, 1]\), then \( f = \tilde{f} + e^{-x} \tilde{f}_1 \), where \( \tilde{f}_1 \) is convergent for \( |x| > R > 0 \).

**Solution:**

Claim that a necessary and sufficient condition for \( \tilde{f} \) to be convergent for \( |x| > R \) is that for any \( \rho > R \),

\[
\sum_{k=0}^{\infty} k! |c_k| \rho^{-k-1} < \infty \tag{1}
\]

Note from Watson’s lemma that

\[
f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} k! c_k x^{-k-1} \tag{2}
\]
Thus if $|x| > R$, we may choose $\rho$ such that $|x| > \rho > R$ and use (1) to conclude (2) is absolutely convergent. Now assume (2) is convergent for any $|x| > R$. Take any $\rho > R$. Choose $x$ so that $\rho > x > R$. From convergence of (2) at $x$, it follows that there exists $M$ so that $|c_k| \leq M x^{k+1}/k!$. Using this, convergence of (1) follows. Thus, part a. is proved.

For part b., if $\tilde{f}$ converges to $f$ for any $x > R$, it follows from analyticity that $\tilde{f} = f$ and so $f \to 0$ as $|x| \to \infty$. From the the integral representation, $f$ is entire and so Liouville Theorem implies that $f = 0$.

For part c., since $\tilde{f}$ is assumed convergent we assume (1) to hold. It follows that $F(p) = \sum_{k=0}^{\infty} c_k p^k$ is entire. Therefore, for $x > R$,

$$
 f(x) = \int_0^1 e^{-px} F(p) dp = \sum_{k=0}^{\infty} \frac{k! c_k}{x^{k+1}} \int_0^x \frac{s^k e^{-s}}{k!} ds
$$

$$
 = \sum_{k=0}^{\infty} \frac{k! c_k}{x^{k+1}} - \sum_{k=0}^{\infty} \frac{k! c_k}{x^{k+1}} \int_1^x \frac{s^k e^{-s}}{k!} ds = \tilde{f}(x) + e^{-x} \tilde{f}_1(x)
$$

where

$$
 \tilde{f}_1(x) = -\sum_{k=0}^{\infty} \frac{k! c_k}{x^{k+1}} \int_0^\infty \frac{(s+x)^k}{k!} e^{-s} ds = -\sum_{k=0}^{\infty} \frac{k! c_k}{x^{k+1}} \left( \sum_{l=0}^{k} \frac{x^l}{l!} \right) = -\sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=l}^{\infty} \frac{c_k k!}{x^{k+1-l}}
$$

which is convergent for $|x| \geq \rho > R$ since from (1),

$$
 |\tilde{f}_1(x)| \leq \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=0}^{\infty} \frac{|c_k| k!}{x^{k+1-l}} \leq \sum_{l=0}^{\infty} \frac{\rho^l}{l!} \left( \sum_{k=0}^{\infty} \frac{|c_k| k!}{\rho^{k+1}} \right) \leq M e^\rho
$$
