1. (Exercise 3.94, page 47, Text). Determine the asymptotics of

\[ I(x) = \int_0^\infty e^{-xp} \cos \frac{1}{p} dp \text{, as } x \to +\infty \]

**Solution:** We note that \( I(x) = \Re I_1(x) \), where

\[ I_1(x) = \int_0^\infty e^{-xp-i/p} dp = x^{-1/2} \int_0^\infty e^{-\sqrt{x}(t+i/t)} dt \equiv x^{-1/2} I_2(x^{1/2}), \]

where

\[ I_2(y) = \int_0^\infty e^{-y(t+i/t)} dy = \int_0^\infty e^{yg(t)} dt \]

We note that \( \Re g(t) = -\Re(t + i/t) \to -\infty \) as \( \Re t \to \infty \). Since \( g' = -1 + i/t^2 = 0 \) at \( t = \pm e^{i\pi/4} \). From \( t = 0 \) the local steepest ascent direction, where \( \Im g = \text{Constant} \) is tangent to the positive \( \Im g \) axis. We pick the value of the constant that passes through the saddle \( t = e^{i\pi/4} \equiv t_s \), i.e. we pick \( \Im g = \Im g(e^{i\pi/4}) = -3 (2e^{i\pi/4}) = -\sqrt{2} \), since we want to join up to \( +\infty \). We note

\[ g(t_s) = -t_s - i/t_s = -2e^{i\pi/4} = -\sqrt{2} - i\sqrt{2} \]

We choose change of variable \( v \), which is real on the steepest line and given by

\[ v^2 = g(t_s) - g(t) = (t + i/t) - t_s - i/t_s = e^{-i\pi/4}(t - t_s)^2 + O(t - t_s)^3 \]

\[ v = e^{-i\pi/8}(t - t_s) + O(t - t_s)^2 + .. \]

So

\[ t - t_s = e^{i\pi/8}v + O(v^2) \equiv \phi(v) \]

So,

\[ I_2(y) = e^{iyg(t_s)} \int_{-A}^\infty e^{-yv^2} \phi'(v)dv, \]

where \( \phi(-A) = -t_s \) and we note that \( v = +\infty \) does correspond to \( t = +\infty \). So, we have

\[ I_2(y) = e^{i\pi/8-\sqrt{y}(1+i)} \left[ \int_{-\infty}^\infty e^{-yv^2} dv \right] (1 + O(y^{-1/2})) = e^{i\pi/8-\sqrt{y}(1+i)} \sqrt{\pi}y(1 + O(y^{-1/2})), \]

i.e.

\[ I_1(x) = e^{i\pi/8-\sqrt{2x^{1/2}(1+i)}x^{-3/4}} \sqrt{\pi}[1 + O(x^{-1/4})] \]

and

\[ I(x) = \sqrt{\pi}x^{-3/4} e^{-\sqrt{2x}} \cos \left( \frac{\pi}{8} - \sqrt{2x} \right) [1 + O(x^{-1/4})] \]
2. Exercise 3.105, Page 51 text. Show that

\[ Ai(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-px + p^3/3} dp \sim \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-2/3 x^{3/2}} [1 + o(1)] \text{ as } x \to +\infty \]

**Solution:** Scaling \( p = x^{1/2} t \), we obtain

\[ Ai(x) = \frac{x^{1/2}}{2\pi i} I(x^{3/2}), \text{ where} \]

\[ I(y) = \int_{-\infty}^{\infty} e^{-y(t^3/3)} dt = \int_C e^{yg(t)} dt \]

We note that \( g'(t) = -1 + t^2 \). So, there are saddle points at \( t = \pm 1 \). As discussed in class, the one relevant to the path \( C \) is the one located at \( t = +1 \) since we are jointing \( \infty e^{i\pi/3} \) with \( \infty e^{-i\pi/3} \). Therefore, we define

\[ v^2 = g(1) - g(t), \text{ i.e. } v^2 = -2/3 + t - t^3/3 = -(t-1)^2 - \frac{1}{3}(t-1)^3, \]

which must be real valued on the steepest descent path through saddle point \( t_s = 1 \). We note on taking square-root, for small \( t - 1 \),

\[ v = -i(t-1) \left[ 1 - \frac{1}{6} (t-1) + \ldots \right] \]

(note the choice of \( \sqrt{-1} = -i \) is made so that the part of the contour \( C \) in the fourth quadrant in \( t \)-plane corresponds to \( v \) positive). So, for small \( v \)

\[ t - 1 = iv + O(v^2) \equiv \phi(v) \text{ implying } \phi'(v) = i + O(v) \]

Therefore,

\[ I(y) = e^{-2y/3} \int_{-\infty}^{\infty} e^{-yv^2} \phi(v) dv = ie^{-2y/3} \left[ \int_{-\infty}^{\infty} e^{-yv^2} dv \right] \left[ 1 + O(y^{-1/2}) \right] = ie^{-2y/3} \sqrt{\frac{\pi}{y}} \left[ 1 + y^{-1/2} \right] \]

Therefore,

\[ Ai(x) = \frac{x^{1/2}}{2\pi i} I(x^{3/2}) = \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-2/3 x^{3/2}} [1 + O(x^{-3/4})] \]

3. Determine the full asymptotic expansion of

\[ I(x) = \int_{-\infty}^{\infty} \cos(tx) e^{-t^2} dt \]

Note that we may rewrite \( I(x) = \Re I_1(x) \), where

\[ I_1(x) = \int_{-\infty}^{\infty} e^{ix - t^2} dt \]
Scale $t = x \tau$. Then,

$$I_1(x) = x \int_{-\infty}^{\infty} e^{x^2(it-t^2)} dt \equiv xI_2(x^2)$$

where

$$I_2(y) = \int_{-\infty}^{\infty} e^{yg(t)} dt, \quad \text{where } g(t) = it - t^2$$

Note, $g'(t) = i - 2t = 0$ at $t = i/2$ and $g(i) = -1/4$. So, we may write

$$I_2(y) = e^{yg(i/2)} \int_{-\infty}^{\infty} e^{-y(g(i/2) - g(t))} dt,$$

For steepest descent path through saddle $t = i/2$, we know $v$ is real, where

$$v^2 = g(i/2) - g(t) = \frac{1}{2}g''(i/2)(t - i/2)^2(t - i/2)^2$$

$$t - i/2 = v \equiv \phi(v)$$

So, $\phi'(v) = 1$, and we obtain

$$I_2(y) = e^{-y/4} \int_{-\infty}^{\infty} e^{-yv^2} dy = \sqrt{\pi y}$$

So, we have

$$I(x) = \Re I_1(x) = \sqrt{\pi} e^{-x^2/4}$$

**Note:** In this case, the result is exact; and we could have avoided the steepest descent type exercise to start off with. But, the problem has been done in a way that is generalizable to

$$\int_{-\infty}^{\infty} e^{ipx - p^2n} dp$$

for any integer $n$. In that case, we would not have $\phi'(v) = 1$, but we could still find the local asymptotic behavior near $v = 0$, which is all that matters for the asymptotic expansion in $x$.

4. Exercise 3.123 (page 55 Text). Assume that the Taylor series coefficient $c_n$ of $f$ at the origin has the asymptotic expansion

$$c_n = an^{-2} + O(n^{-3})$$

Show that $f$ is singular necessarily on the unit circle at $z = 1$.

**Solution:** As noted in the hint, consider

$$g(z) = f'(z) - \sum_{n=2}^{\infty} \frac{a}{n-1} z^{n-1} = c_1 + \sum_{n=2}^{\infty} n \left( c_n - \frac{a}{n(n-1)} \right) z^{n-1}$$
From given condition, on $c_n$, since for large $n$, $c_n - \frac{a}{n(n-1)} = c_n - \frac{a}{n^2} + O(n^{-2}) = O(n^{-2})$, the series for $g(z)$ is absolutely convergent on $|z| = 1$ since $\sum \frac{1}{n^2} < \infty$. Therefore, $g(z)$ is continuous in the closure of the unit circle. On the other hand, note that

$$a \sum_{n=2}^{\infty} \frac{z^{n-1}}{n-1} = -a \ln(1 - z)$$

which is singular on $|z| = 1$ exactly at $z = 1$. Therefore, it follows that $f'$ blows up on $|z| = 1$ only at $z = 1$. 