Steepest descent method

Remark: This method is suitable for analyzing

$$I(x) \ = \ \int_C e^{-xp(t)} \ q(t) \ dt \tag{1}$$

where the path C is in the complex t plane. It could either be open or closed, of finite length or otherwise (the integral has to exist, however). $p(t)$ and $q(t)$ are both analytic functions of t, except perhaps at some singular points. x will be taken to be real and positive, at least for now. The interest is in determining the asymptotic behavior of $I(x)$ for large x. Since $p(t)$ is complex, Laplace's method or integration by parts cannot be applied directly. However, if it is possible to deform contour C into a single contour C_1, defined by

$$Im \ p \ = \ constant \tag{2}$$

then we note that (1) becomes an integral of the form

$$I(x) \ = \ e^{-i \ x \ Im \ p} \int_a^b e^{-x \ Re \ p(t(\tau))} \ q(t(\tau)) \ t'(\tau) \ d\tau \tag{3}$$

where $t(\tau)$ is a chosen parametrization of the path C_1, where τ is real and ranges from $(a, \ b)$ (we will assume $t'(\tau)$ is well defined and nonzero on C_1). Such a path is called a steepest descent path. In the form (3), one can use Laplace's method or integration by parts to determine the asymptotic behavior of $I(x)$ for large x. It is to be noted that if such a path includes no point where $p(t)$ is singular or $p'(t)$ zero, then $Re \ p(t(\tau))$ must be monontonically increasing or decreasing function of τ. To prove this claim, we note that if it were otherwise then there would be a point $\tau = \tau_0$, where

$$\frac{d}{d\tau} \ Re \ p(t(\tau)) \ = \ 0 \ = \ Re \ \left[p'(t(\tau_0)) \ t'(\tau_0) \ \right] \tag{4}$$

On the otherhand, from differentiating (2) with respect to τ, it follows that $Im \ [p'(t(\tau_0)) \ t'(\tau_0)] \ = \ 0$ Together, with (4), this implies that $p'(t(\tau_0)) \ = \ 0$, since $t'(\tau_0) \ \neq \ 0$. This proves that, except for critical points on the contour C_1 (where p' is either singular or zero), $Re \ p$ is indeed monotonic (Hence the name steepest descent path).

Comment: Generally, it is not possible to have just one steepest descent path C_1 equivalent to the original path C. This will be so if $Im \ p$ takes differing values at the end points of C. Even when they take the same value, there may be no direct steepest descent path connecting the two end points. Two or more steepest descent path are usually equivalent to the original C. We refrain from discussing the most general situation, as this can be quite complicated. We will only illustrate the situation through examples.

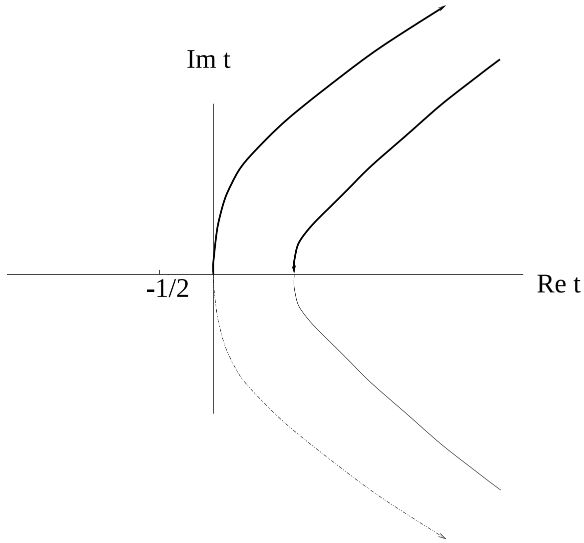

Figure 1: Steepest descent paths for Eq. (5)

Example 1:

$$I(x) \;=\; \int_0^1 \frac{1}{\sqrt{t}}\, exp\left[ix(t \;+\; t^2\right]\; dt \tag{5}$$

Want to determine the asymptotics of $I(x)$ as $x \;\rightarrow\; +\infty$.
Solution: In this case

$$p(t) \;=\; -\,i\;(t \;+\; t^2) \tag{6}$$

We note that

$$Im\; p(0) \;=\; 0 \;\neq\; Im\; p(1) \;=\; -\,2 \tag{7}$$

So, there isn't a single steepest descent path connecting $t \;=\; 0$ to $t \;=\; 1$. To determine how to get from one end point to the other, using steepest descent paths, we need to consider set of curves defined by the relation

$$Im\; p(t) \;=\; -\;\; Re\;(t \;+\; t^2) \;=\; c \;\;\text{, a constant} \tag{8}$$

If we write $t \;=\; \xi \;+\; i\eta$, the above implies

$$\xi \;+\; \xi^2 \;-\; \eta^2 \;=\; (\xi \;+\; 1/2)^2 \;-\; \eta^2 = \;-\,c \;+\; 1/4 \tag{9}$$

This is a set of hyperbola. We note that for $c \;=\; 0$ and $c = -2$, the two hyperbola pass through $t \;=\; 0$ and $t \;=\; 1$ respectively (See Fig. 1). However, on inspection it is clear on each of these parabola, $Re\; p$ increases monotonically to $+\infty$, only if we go towards $\infty\; e^{i\pi/4}$, since $Re\; p \;\sim\; Re\;(-it^2)$ for large t. The monotonicity follows from the fact that p' is only zero at $t \;=\; -\,\frac{1}{2}$ that is not on either of the hyperbola of interest. Thus, we deform the original real line path in the t-plane to steepest descent paths C_1 and C_2, shown in Fig. 1. We note that while paths C_1 and C_2 never join up, at $\infty\, e^{i\pi/4}$, the integrand $|e^{-xp(t)}| = e^{-x\ Re\ p}$ tends to zero exponentially and there is no contribution on a connecting path from C_1 to C_2 at a distance R from the origin, in the limit $R \to \infty$. Thus, one single original integral becomes equivalent to sum of two different steepest descent paths C_1 and C_2. Now to obtain the contribution from C_1, we note that if we define

$$\tau = -i\,(t + t^2) \tag{10}$$

then τ is real along C_1 (because of (8)) and varies from 0 to ∞ on C_1. Further,

$$t = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4i\,\tau} \tag{11}$$

and

$$t'(\tau) = \frac{i}{\sqrt{1 + 4i\,\tau}} = i\,\left(1 - 2\,i\,\tau + b_2\,\tau^2 + ...\right) \tag{12}$$

for some constants b_2, b_3..., etc. that can be readily calculated. Also,

$$\frac{1}{\sqrt{t}} = e^{-i\pi/4}\ \ \tau^{-1/2}\left[1 + c_1\,\tau + c_2\,\tau^2 + ..\right] \tag{13}$$

Thus,

$$\frac{1}{\sqrt{t(\tau)}}\,t'(\tau) \sim e^{i\pi/4}\,\tau^{-1/2}\left[1 + d_1\,\tau + d_2\,\tau^2 + ...\right] \tag{14}$$

where d_1, d_2, etc. can be readily calculated. Thus, from Watson's Lemma:

$$\int_{C_1}\frac{1}{\sqrt{t}}\,exp\left[ix(t + t^2)\right] \sim e^{i\pi/4}\sum_{j=0}^{\infty}\frac{d_j\,\Gamma(j+1/2)}{x^{j+1/2}} \tag{15}$$

where d_0 is defined to be 1.

Now, on C_2, it is convenient to define

$$\tau = p(t) - p(1) = -i(t^2 + t - 2) \tag{16}$$

This is real and varies from 0 to ∞, as we away from $t = 1$ on steepest descent path C_2. In this case, it is possible to explicitly invert (16) to get

$$t(\tau) = -\frac{1}{2} + \frac{3}{2}\sqrt{1 + \frac{4i\tau}{9}} \tag{17}$$

and so

$$t'(\tau) = \frac{i}{\sqrt{9 + 4i\tau}} = \frac{i}{3}\left[1 + a_1\,\tau + a_2\,\tau^2 + \ \right] \tag{18}$$

and

$$t^{-1/2}(\tau) \;=\; 1 \;+\; b_1\tau \;+\; ... \tag{19}$$

The product

$$t'(\tau)\; t^{-1/2}(\tau) \;\sim\; \frac{i}{3}\; \left[1 \;+\; c_1\;\tau \;+\; c_2\;\tau^2 \;+\; ..\right] \tag{20}$$

where the coefficients c_1, c_2, etc. are possible to calculate readily from a_j's and b_j's in (18) and (19). Thus,

$$\int_{C_2} \frac{1}{\sqrt{t}}\; exp\;[ix\;(t^2 \;+\; t)]\; dt \;=\; -\,e^{2\;i\;x} \int_0^\infty d\tau\; \frac{t'\tau}{\sqrt{t(\tau)}}\; e^{-x\tau} \tag{21}$$

Applying Watson's Lemma and the known asymptotic expansion (20) , we get

$$\int_{C_2} \frac{1}{\sqrt{t}}\; exp\;[ix\;(t^2 \;+\; t)]\; dt \;\sim\; \frac{i}{3}\; e^{2ix} \sum_{j=0}^{\infty} \frac{c_j\; j!}{x^{j\;+\;1}} \tag{22}$$

where c_0 is defined as 1. By adding (22) to (15), we obtain the complete asymptotic expansion for $I(x)$. The contributions come from the end points; this is not unexpected since the integrand decrease exponentially away from the end points 0 and 1. Interestingly, the leading order $O(x^{-1/2})$ contribution comes only from the end point $t \;=0$. This is because, even though the exponential is of the same order, $q(t) \;=\; t^{-1/2}$ is singular at $t \;=\; 0$, while it is not at $t \;=\; 1$. The result (22) itself could have been alternately be obtained merely by integration by parts, since $q(t)$ is analytic at $t \;=\; 1$; the same is not true for the contribution (15).

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Steepest descent with saddle points:

Remark: In studying the asymptotics of

$$I(x) \;=\; \int_C \; e^{-x\;p(t)}\; q(t)\;\; dt \tag{1}$$

for large x, sometimes in following a steepest descent path, one is forced to pass through a critical point, where $p'(t)$ is either singular or zero. If the latter is the case, we call such a point a saddle point. The name comes from the fact that if $p'(t_0) \;=\; 0$, then in a neighborhood of t_0, *Re* p is a harmonic function with a saddle at $t = t_0$ (t_0 cannot be a maximum or minimum because of maximum principle). The simplest kind of saddles are those for which $p'(t_0) \;=\; 0$, but $p''(t_0) \;\neq\; 0$. An m-th order saddle is one for which all derivatives upto m-th order are zero, but the $(m+1)$-th derivate is nonzero at t_0. When a steepest descent path goes through a saddle t_0, if *Re* p is smaller at t_0 than at the end points, the asymptotic expansion of the integral is completely determined by the contribution from a small neighborhood of t_0. When multiple saddles are involved, the contribution from each has to be generally considered unless $Re\ p$ at one saddle is smaller than $Re\ p$ from any other saddles or end points. In the following, we illustrate saddle point contribution in a steepest descent analysis:

Example 1: Determine the asymptotic behavior of $I(x)$ as $x \to \infty$, where

$$I(x) = \int_{-\infty}^{\infty} exp\left[ix \left(t^3/3 + t\right)\right] \tag{2}$$

Comment: The above function is related to the Airy function, which is a solution to the differential equation

$$y'' - x\, y = 0 \tag{3}$$

called Airy's equation. The particular form of solution (2) can be arrived at more systematically in the following manner. We look for solution with a Fourier-representation

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\ x\ k}\ Y(k)\ dk \tag{4}$$

We note that

$$y''(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} k^2\ e^{i\ x\ k}\ Y(k)\ dk \tag{5}$$

$$x\ y = = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{i\ x\ t}\ Y'(k)\ dk$$

assuming that these integrals exist. Therefore, on Fourier transforming (3), we obtain

$$-i\ Y'(k) - k^2\ Y(k) = 0 \tag{6}$$

Thus,

$$Y(k) = \text{Constant}\ e^{i\ k^3/3} \tag{7}$$

On taking the constant in (7) to be unity, and substituting into (4), we obtain

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\ x\ [k\ +\ k^3/3]}\ dk \tag{8}$$

This is the Airy function, usually denoted by $Ai(x)$. If we now substitute $k = x^{1/2}\ t$ into (8), we obtain:

$$Ai(x) = \frac{x^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{i\ x^{3/2}\ [t\ +\ t^3/3]}\ dt = \frac{x^{1/2}}{2\pi}\ I(x^{3/2}) \tag{9}$$

Solution to example 1:

We note that

$$p(t) = -\,i\ (t + t^3/3) \tag{10}$$

Now, we seek to connect $t = -\infty$ to $t = +\infty$ on one or more steepest descent paths. We notice that

$$p'(t) = -i\,(1 + t^2) = 0 \text{ at } t_0 = \pm i$$

However, clearly p'' is nonzero at $\pm i$. We note that at both saddles,

$$Im\ p = 0 \tag{11}$$

Also, we note that $Im\ p = 0$ on the entire imaginary axis. Also,

$$Re\ p(i) = \frac{2}{3} \quad \text{while} \quad Re\ p(-i) = -\frac{2}{3} \tag{12}$$

Now, in a neighborhood of $t = i$, from the Taylor expansion of p, we get

$$p(t) = \frac{2}{3} + (t-i)^2 - \frac{i}{3}(t-i)^3 \tag{13}$$

If we write $t - i = r_1\, e^{i\theta_1}$, then (11) implies that for small r_1,

$$Im\ p = r_1^2\ sin\ 2\theta_1 + O(r_1^3) = 0 \quad \text{for} \quad \theta_1 = n\,\pi/2 \tag{14}$$

for integral n. Also, it is clear that $Re\ p$ increases outwards as r_1 increases, when n is even. Thus, the local descent directions (i.e. directions in which the integrand in (1) decreases in size) is given by $\theta_1 = 0$ and $\theta_1 = \pi$. Now, the steepest descent direction from $t = i$, that for small r_1 is directed towards $\theta_1 = 0$ must approach one of the descent directions at ∞.

Now consider large $|t|$. Putting $t = r\, e^{i\theta}$, we get for large r,

$$p(t) \sim -i\, t^3/3 = -\frac{i}{3} r^3\, e^{i\,3\,\theta} \quad \text{meaning} \quad Im\ p = 0 \quad \text{for} \quad \theta = \pi/6 + n\,\pi/3 \tag{15}$$

for integral n. Among these, only the even n values correspond to descent, since $Re\ p$ increases to ∞ as $r \to \infty$ along those directions. Thus $\theta = \pi/6, 5\pi/6, -\pi/2$ are the local descent directions.

Thus, the descent curve emanating from $t = i$, towards the first quadrant, can either approach $r \to \infty$ with $\theta = \pi/6$, $\theta = 5\pi/6$ or $\theta = -\pi/2$ since on the descent path, $Re\ p(t)$ monotonically increases to ∞. We will now argue that it must be $\theta = \pi/6$: First, this steepest descent path cannot cross the imaginary t-axis segment $[0, i\infty)$, since on this segment, $Re\ p(t) < Rep(i)$, while $Re\ p$ on the steepest descent path increases monotonically. Second, this steepest descent path cannot cross positive $Re\ t$-axis, since on the positive real axis $Im\ P(t) = -(t + t^3/3) < 0$. Thus, the descent path must approach $r \to \infty, \theta = \pi/6$.

Similar analysis near $t = -i$, shows

$$p(t) = -\frac{2}{3} - (t+i)^2 - \frac{i}{3}(t+i)^3 \tag{16}$$

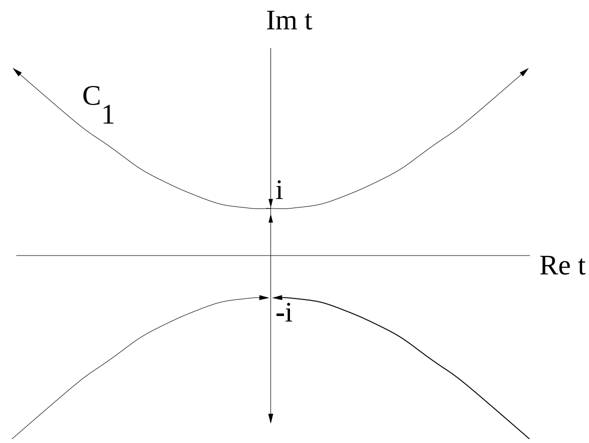

Figure 2: Steepest descent path C_1 for Eq. (2)

and so the descent directions towards $arg\ (t+i) \ = \ \pm\,\pi/2$, while the ascent directions are locally towards $arg\ (t+i) \ = \ 0, \ \ \pi$. The local ascent direction from $t \ = \ -\,i$ in the fourth quadrant must approach $r = \infty$, $\theta \ = \ -\,\pi/6$, since it cannot cross the positive real axis (where $Im\ p \ \neq \ 0$) and negative imaginary axis (where $Rep(t) \ > \ Rep(-i)$).

From symmetry, about the imaginary real axis, it follows that the set of all curves given by $Im\ p \ = \ 0$ is given by Fig. 1. It is now to be noted from contour integration that

$$I(x) \ = \ \int_{C_1} \ e^{ix(t \ + \ t^3/3)} \ dt \tag{17}$$

where C_1 is the steepest descent path connecting $\infty \ e^{i5\pi/6}$ to $\infty \ e^{i\pi/6}$ Thus, the asymptotics of $I(x)$ is dominated by the contribution from the saddle $t \ = \ i$. It is to be noted that even though $Re\ p$ is smaller at the other saddle $t \ = \ -\,i$, there is no contribution from this since the steepest descent path equivalent to the original contour integral does not pass through $t \ = \ -\,i$.

We now want to evaluate use (17) to find the asymptotics for large $|x|$. We break up the contour C_1 into $C_{1,1}$ and $C_{1,2}$, corresponding to left and right of $t = i$. On $C_{1,2}$, define

$$\tau \ = \ p(t) - \frac{2}{3} \ = \ (t-i)^2 \ - \ \frac{i}{3}\ (t-i)^3 \tag{18}$$

Then,

$$t - i \ = \ \sqrt{\tau} \ + \ a_1\ \tau \ + \ a_2\ \tau^{3/2} \ + \ .... \tag{19}$$

where a_1, a_2 and other such coefficients can be found readily by substituting (19) into (18) and equating powers of τ on both sides of the equation. From (19),

$$t'(\tau) \ = \ \frac{1}{2\sqrt{\tau}} \ + \ a_1 \ + \ \frac{3}{2}\ \tau^{1/2} \ + \ ... \tag{20}$$

Thus,

$$\int_{C_{1,2}} dt\ e^{ix(t+t^3/3)} \ = \ \int_0^\infty d\tau\ t'(\tau)\ e^{-x\tau} \ \sim \ \sum_{j=0}^{\infty} \frac{(j+1)a_j}{2x^{(j+1)/2}}\ \Gamma\ [(j+1)/2] \tag{23}$$

For the contour segment $C_{1,1}$, the sign of $\sqrt{\tau}$ occurring in (19) has to be reversed. Further, the limit of integration in (23) in the τ variable has to be ∞ to 0. Thus,

$$\int_{C_{1,1}} dt\ e^{ix(t+t^3/3)} \ = \ \int_\infty^0 d\tau\ t'(\tau)\ e^{-x\tau} \ \sim \ \sum_{j=0}^{\infty} (-1)^j\ \frac{(j+1)a_j}{2x^{(j+1)/2}}\ \Gamma\ [(j+1)/2] \tag{24}$$

where $a_0 \ = \ 1$. Combining (23) and (24), we get

$$I(x) \ = \ \int_{C_1} dt\ e^{ix(t+t^3/3)} \ \sim \ \sum_{j=0}^{\infty} \frac{(2\ j+1)a_{2\ j}}{x^{j+1/2}}\ \Gamma\ [j+1/2] \tag{25}$$

Large $|x|$ asymptotics for complex x: Stokes phenomena

Problem: We determined the asymptotic behavior of $I(x)$, defined as:

$$I(x) \ = \ \int_{-\infty}^{\infty} dt\ exp\left[ix\ \left(t^3/3\ +\ t\right)\right] \tag{26}$$

for x large and positive. We now want to consider x complex. First, the integral as defined in (26) does not make sense for complex x. However, through contour deformation, it is clear that the analytic continuation of $I(x)$ off the real axis is determined by

$$I(x) \ = \ \int_{\infty\ e^{i5\pi/6-i\phi/3}}^{\infty\ e^{i\pi/6-i\phi/3}} dt\ exp\left[ix\ \left(t^3/3\ +\ t\right)\right] \tag{27}$$

where

$$x \ = \ |x|\ e^{i\phi} \tag{28}$$

We now want to consider the asymptotics for $|x| \ >> \ 1$ for nonzero ϕ.

Solution: First we consider $\pi \ > \ \phi \ > \ 0$. We note that (27) can be written in more standard asymptotic form

$$\int_C e^{-|x|\ p(t)}\ dt \tag{29}$$

for which we applied the steepest descent method, provided we now take

$$p(t) \ = \ -\,e^{i\phi}\ (t^3/3\ +\ t) \tag{30}$$

As before the saddle points are at $t \ = \ \pm\, i$. Near the saddle point $t \ = \ i$,

$$p(t) \ = \ -\,e^{i\phi}\ (\frac{2}{3}\ +\ (t-i)^2\ +\ ..) \tag{31}$$

Therefore, if $t - i \; = \; r_1 \; e^{i\theta_1}$, then for $r_1 \; << \; 1$,

$$Im \; p \; = \; Im \; p(i) \; = \; \frac{2}{3} \; sin \; \phi \quad \text{implying} \quad sin \; (2 \; \theta_1 \; + \; \phi) \; = \; 0 \tag{32}$$

Since

$$Re \; p \; = \; \frac{2}{3} \; cos \; \phi \; + \; r_1^2 \; cos \; (2\theta_1 \; + \; \phi) \; + \; O(r_1^3) \tag{33}$$

It follows that for even n, the local directions from $t = i$, corresponding to

$$\theta_1 \; = \; n \; \pi/2 \; - \; \phi/2 \tag{34}$$

correspond to descent, while odd n correspond to ascent. Notice that corresponding to x real and positive, i.e. $\phi = 0$, the local descent and ascent paths have rotated in a clockwise direction by $\phi/2$.

Similar analysis near $t = -i$ shows that for $t + i \; = \; r_2 \; e^{i\theta_2}$, in the limit $r_2 \; << \; 1$, the descent directions from $t = -i$ correspond to

$$\theta_2 \; = \; n \; \pi/2 \; - \; \phi/2 \tag{35}$$

for odd n, while even n correspond to ascent. Once again the local descent and ascent directions have been rotated clockwise by $\phi/2$, compared to the direction for $\phi = 0$.

Now for large t,

$$p(t) \; \sim \; - \, i \; e^{i\phi} \; t^3/3 \tag{36}$$

So, if $t \; = \; r \; e^{i\theta}$, then for $r \; >> \; 1$,

$$Im \; p \; = \; \text{constant} \quad \text{corresponds to} \quad - \, \frac{r^3}{3} \; cos \; (\phi \; + \; 3\theta) \; \sim \; \text{Constant}$$

i.e.

$$\theta \; = \; - \, \phi/3 \; + \; n \; \pi/3 \; + \; \pi/6 \tag{37}$$

Since for large r,

$$Re \; p \; \sim \; \frac{r^3}{3} \; sin \; (\phi \; + \; 3 \; \theta) \tag{38}$$

the even values of n in (38) correspond to descent paths, while odd values correspond to ascent. Note that compared to $\phi \; = \; 0$, (38) implies that ascent and descent directions have rotated clockwise by $\phi/3$.

Now the descent path emanating from $t \; = \; i$, corresponding locally to $\theta_1 \; = \; - \, \phi/2$ must approach one of the three descent paths at ∞. First we rule out this descent path intersecting the imaginary axis at all. On the imaginary axis, $t \; = \; iv$, and so

$$Im \; p \; = \; (v \; - \; v^3/3) \; sin \; \phi \; = \; Im \; p(i) \; = \; \frac{2}{3} \; sin \; \phi \tag{15}$$

only when $v = 1$ (double root of cubic) and $v = -2$. Thus, the descent path in question can only intersect the imaginary axis at $t = -2i$ (other than $t = i$, where it originated). However,

$$Re\ p \ = \ (v - v^3/3)\ cos\ \phi \tag{41}$$

and

$$Re\ p(-2i) \ = \ \frac{2}{3}\ cos\ \phi \ = \ Re\ p(i) \tag{42}$$

The equality (42) rules out this descent path from passing from $t \ = \ -2i$, since on a descent path $Re\ p$ must be monotonically increasing, except when it passes through a saddle or a nonanalytic point. Thus this descent path must approach $\infty\ e^{i\pi/6-\phi/3}$; others are not even on the right half t plane.

Similar reasoning applied to the descent path, locally directed as $\theta_2 \ = \ \pi/2 - \phi/2$ (coming out of $t \ = \ -i$), shows that it cannot possibly intersect the imaginary axis and must therefore asymptote to the same descent direction $\infty\ e^{i\ (\pi/6-\phi/3)}$.

Now consider the descent path away from $t = i$, corresponding to $\theta_1 \ = \ \pi \ - \ \phi/2$. At large distances, this must approach one of the two descent paths $\infty\ e^{i5\pi/6-\phi/3}$ or $\infty\ e^{-i\pi/2-\phi/3}$ on the left half t-plane, since it cannot intersect the imaginary axis once again. However, on the real negative t axis,

$$Re\ p \ = \ sin\ \phi\ (t \ + \ t^3/3) \tag{43}$$

is negative and therefore since $Re\ p(i) \ = \ \frac{2}{3}\ cos\ \phi$ is non-negative for $0 \ < \ \phi \ \leq \ \pi/2$, there is no chance of this steepest descent path intersecting the negative real axis. For $\pi/2 \ < \ \phi \ < \ \pi$, we note that while $Im\ p(i) \ = \ \frac{2}{3}\ sin\ \phi \ > \ 0$, on the negative real axis

$$Im\ p \ = \ - \ cos\ \phi\ (t \ + \ t^3/3) \ > \ 0 \tag{44}$$

Thus, in this case as well, there is no possibility of this descent path to cross the negative real axis. Therefore, we are forced to conclude that for large $|t|$, this descent path asymptotes $\infty\ e^{i5\pi/6-\phi/3}$.

Similar considerations for all other steepest descent paths emanating from saddle points $t \ = \ \pm\, i$ leads us to Fig. 2.

As far as evaluating the answer, the only relevant information is that the steepest descent path passes through $t = i$ that is locally directed towards $\theta_1 \ = \ -\,\phi/2$ and $\theta_1 \ = \ \pi \ - \ \phi/2$. To calculate the leading order answer, we can resort to Laplace's method and write down

$$I(x) \ \sim \ exp\,[-\frac{2}{3}e^{i\phi}|x|]\ \left\{ \int_0^{\epsilon} e^{-i\ \phi/2}\ dr_1\ e^{-|x|\ r_1^2} \ - \ \int_{\epsilon}^{0} e^{-i\ \phi/2}\ dr_1\ e^{-|x|r_1^2} \right\} \ \sim \ exp[-\frac{2}{3}x]\sqrt{\frac{\pi}{x}} \tag{45}$$

Now, we consider the special case $\phi \ = \ \pi$. At this point, the steepest descent path looks like that shown in Fig. 3 (Leave it upto you to confirm that this is the case). Note that

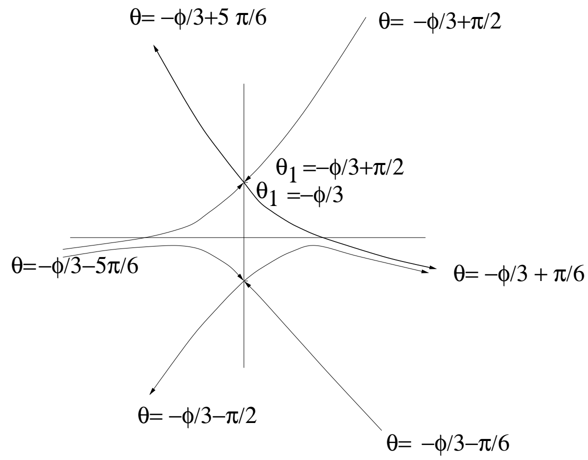

Figure 3: Steepest descent path for $0 < \phi < \pi$. The one relevant for integration shown in thicker line

in evaluating the asymptotics of $I(x)$, we now collect contribution from both the saddles. However, since

$$Re\ p(-i) \ = \ -\frac{2}{3}\ cos\ \phi \ > \ Re\ p(i) \ = \ \frac{2}{3}\ cos\ \phi \ \ \text{for} \ \ \phi \ = \ \phi \tag{46}$$

The contribution from the saddle $t \ = \ -i$ is exponentially small, and the same result (45) is still valid.

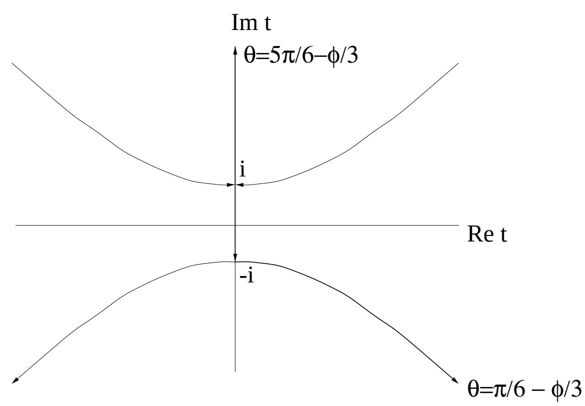

Figure 4: Steepest descent path for $\phi \ = \ \pi$. The one relevant for integration shown in thicker line

For $2\ \pi \ > \phi \ > \ \pi$, the steepest descent paths change qualitatively as shown in Fig. 3 (This can be argued using the same kind of consideration, as given above). Now, we get the full contribution from both saddles, and one obtains besides (45) the additional contribution

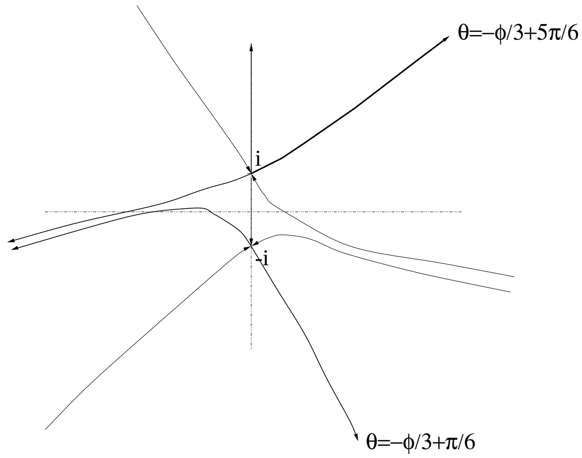

Figure 5: Steepest descent path for $\pi < \phi < 2\,\pi$. The one relevant for integration shown in thicker line

at the saddle $t = -i$:

$$\sim\ exp\,[\frac{2}{3}e^{i\phi}|x|]\ \left\{\int_0^{\epsilon} e^{-i\ \phi/2+i\ \pi/2}\ dr_2\ e^{-|x|\ r_2^2}\ +\ \int_{\epsilon}^{0} e^{-i\ \phi/2\ -\ i\ \pi/2}\ dr_2\ e^{-|x|\ r_2^2}\right\}\ \sim\ i\ exp[\frac{2}{3}x]\sqrt{\frac{\pi}{x}}$$

Combining with (45), we get to the leading order,

$$I(x)\ \sim\ \sqrt{\frac{\pi}{x}}\ \left\{\ exp[-\frac{2}{3}x]\ +\ i\ exp[\frac{2}{3}x]\right\} \tag{47}$$

We note that as long as $\phi\ <\ 3\pi/2$, the result (45) remains valid; at $\phi\ =\ 3\pi/2$, both the exponential contributions are of the same order. For $2\ \pi\ >\ \phi\ >\ 3\pi/2$, only the contribution from the second term is relevant, since the first is exponentially small.

Comment: The change of asymptotic behavior of an analytic function of x for large x, as $\phi\ =\ arg\ x$ is changed is referred to as Stokes phenomena. It seems mysterious, since the analytic function cannot have any discontinuity, yet the formula describing its asymptotic behavior does have an apparent discontinuity across particular values of ϕ, called Stokes lines. This is important in a number of physical applications.