Week 10 Notes, Math 8610, Tanveer

1. Exact Solution Methods for Free Boundary in 2-D Stokes Flow

We will consider the evolution of a drop in a highly viscous fluid driven by surface tension and a source or a sink. In the singly connected smooth domain Ω occupied by the drop, in the formal limit of Navier-Stokes equation as Reynolds number $\frac{1}{\nu} \to 0$, we obtain the steady Stokes approximation[1] Rescaling pressure appropriately, and representing velocity $\mathbf{u} = (u, v)$ in Cartesian coordinates (x, y), we obtain in the Stokes limit:

$$-\nabla p + \Delta(u, v) \; = \; 0 \; ; \; \nabla \cdot (u, v) = 0 \text{ in } \Omega \tag{1.1}$$

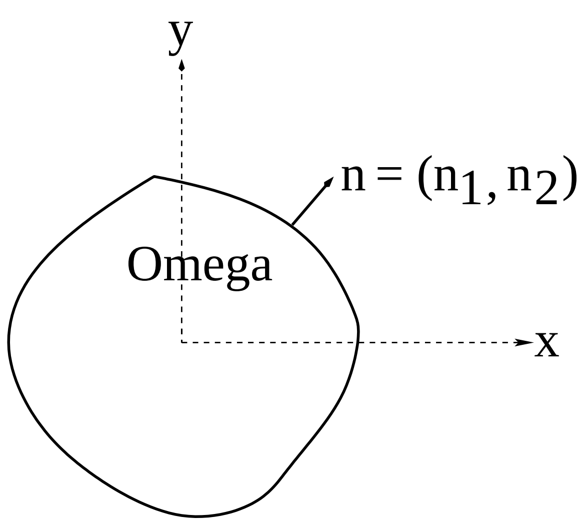

FIGURE 1. Domain $\Omega \subset \mathbb{R}^2$, with normal $\mathbf{n}$ at $\partial\Omega$ and source at $(0, 0)$

For simplicity, we will assume Ω to a singly connected domain (see Fig. 1), though some of the results are valid for multiply connected domain as well. The continuity of stress of at the free boundary and the condition that free boundary moves with the fluid implies

$$-p n_j + 2 S_{jk} n_k = \sigma \kappa n_j \; , \; V_n = (u, v) \cdot \mathbf{n}, \text{ on } \partial\Omega \tag{1.2}$$

where $\mathbf{n}$ is the interface normal, V_n is the interface normal speed, κ the curvature and the strain tensor $\mathbf{S}$ has cartesian components

$$S_{jk} = \frac{1}{2} \left(u_{j,k} + u_{k,j} \right) \tag{1.3}$$

[1] Though $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is time-dependent since $\Omega(t)$ depends on t, the steady Stokes flow approximation is justified when of evolution of interface occurs on a far shorter time scale then evolution of $\mathbf{u}$.

where we identify $\mathbf{u} = (u_1, u_2) = (u, v)$. We note that in terms of (u, v) and (x, y), using divergence condition,

$$S_{11} = u_{1,1} = u_x \ , \ S_{22} = u_{2,2} = v_y = -u_x = -S_{11} \tag{1.4}$$

$$S_{12} = S_{21} = \frac{1}{2}[u_{2,1} + u_{1,2}] = \frac{1}{2}(u_y + v_x) \tag{1.5}$$

We will also allow for the fluid to have a source at $(x, y) = (0, 0)$ assumed to be in Ω; *i.e.* as $(x, y) \to (0, 0)$,

(1.6)

$$(u, v) = \frac{-m}{2\pi(x^2 + y^2)}(x, -y) + O(1), \text{ implying } u - iv \sim -\frac{m}{2\pi z} + O(1) \text{ as } z = x + iy \to 0$$

2. Complex variable formulation of 2-D Stokes flow

Since the flow is two dimensional, the velocity is expressed in terms of Stream function ψ: $(u, v) = (\partial_y \psi, -\partial_x \psi)$; hence scalar vorticity

$$\omega = \partial_x v - \partial_y u = -\Delta \psi \tag{2.7}$$

Applying the divergence operator $\nabla \cdot$ to (1.1), and using divergence free condition on velocity, we obtain

$$\Delta p = 0 \tag{2.8}$$

Further, eliminating pressure p in (1.1), we obtain

$$\Delta(\partial_y u - \partial_x v) = 0 = \Delta \omega \tag{2.9}$$

So, $(p, -\omega)$ forms a harmonic pair of functions. We now show that they form harmonic conjugates since (1.1) gives the Cauchy Riemann conditions for $(p, -\omega)$:

$$\partial_x p = \Delta u = \Delta \partial_y \psi = \partial_y \Delta \psi = -\partial_y \omega \tag{2.10}$$

and

$$\partial_y p = \Delta v = -\Delta \partial_x \psi = \partial_x \omega \tag{2.11}$$

It follows that $p - i\omega$ is an analytic function of $z = x + iy$. We define analytic function $f(z)$ so that

$$p - i\omega = 4f'(z) \tag{2.12}$$

In particular this implies

$$p = 2f'(z) + 2\bar{f}'(\bar{z}) \tag{2.13}$$

$$\omega = 2i\left[f'(z) - 2\bar{f}'(\bar{z})\right], \tag{2.14}$$

where we define analytic function $\bar{h}$ corresponding to h so that for z real,

$$\bar{h}(x) = [h(x)]^* \ \textbf{Note if } h(z) = \sum_{n=0}^{\infty} h_n z^n, \ \bar{h}(z) = \sum_{n=0}^{\infty} h_n^* z^n \tag{2.15}$$

Note that for $z = x + iy$ with real x and y, $[h(z)]^* = \bar{h}(\bar{z})$. We define ϕ so that

$$\Delta \phi = p \tag{2.16}$$

Since $-\Delta \psi = \omega$, it follows that

$$\Delta(\phi + i\psi) = 4f'(z) \tag{2.17}$$

At this point, it is useful to think of (x, y) as complex variables. Then it is possible to think of $z = x + iy$ and $\bar{z} = x - iy$ as two independent variables. Then by using change $(z, \bar{z})$ as independent variables rather than (x, y), it is easy to check using chain rule that

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right)$$
$$\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

which on addition and subtraction gives rise to

(2.18) $$\partial_x = \partial_z + \partial_{\bar{z}}\ ,\ \ \partial_y = i\left(\partial_z - \partial_{\bar{z}}\right)$$

So,

$$\Delta = \partial_x^2 + \partial_y^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

It follows from (2.17) that

(2.19) $$4\frac{\partial^2}{\partial z \partial \bar{z}}\left(\phi + i\psi\right) = 4f'(z)$$

Integration gives rise to

(2.20) $$\phi + i\psi = \bar{z}f(z) + g(z)$$

for analytic functions f and g In particular, using (2.14),

(2.21) $$-\Delta\psi = -4\frac{\partial^2\psi}{\partial z \partial \bar{z}} = \omega = 2i\left[f'(z) - \bar{f}'(\bar{z})\right]$$

So, integration gives rise to

(2.22) $$\psi = -\frac{i}{2}\left[\bar{z}f(z) - z\bar{f}(\bar{z}) + g(z) - \bar{g}(\bar{z})\right]$$

For real (x, y), this gives rise to $\psi(x, y) = \mathrm{Im}\left[(x - iy)f(x + iy) + g(x + iy)\right]$, which is a general representation of solutions to biharmonic equation $\Delta^2\psi = 0$ on the plane. Using,

(2.23)
$$u + iv = \partial_y\psi - i\partial_x\psi = -i\left(\partial_x + i\partial_y\right)\psi = -\partial_{\bar{z}}\left\{\bar{z}f(z) - z\bar{f}(\bar{z}) + g(z) - \bar{g}(\bar{z})\right\},$$

it follows that

(2.24) $$u + iv = -f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z})$$

Also, since $S_{11} = \partial_x u = \frac{1}{2}\left[\partial_x u - \partial_y v\right]$ and $S_{12} = \frac{1}{2}\left[\partial_x v + \partial_y u\right]$, we can easily check

(2.25) $$S_{11} + iS_{12} = \partial_{\bar{z}}\left[u + iv\right] = z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z}),$$

Combining the two implied scalar equations corresponding to $j = 1$ and 2 in the stress equation in (1.2), we obtain

(2.26) $$-p(n_1 + in_2) + 2\left[S_{11}n_1 + S_{1,2}n_2\right] + 2i\left[S_{21}n_1 + S_{22}n_2\right] = \sigma\kappa(n_1 + in_2),$$

Using $S_{11} = -S_{22}$ and $S_{12} = S_{21}$ the above equation reduces to

(2.27) $$-p(n_1 + in_2) + 2\left[S_{11} + iS_{12}\right](n_1 - in_2) = \sigma\kappa(n_1 + in_2),$$

Using $n_1 + in_2 = iZ_s$, $\kappa = \theta_s = [Z_{ss}]/Z_s$, where arclength s increases in the counter-clockwise direction, using (2.13) and (2.25) in (2.27) implies on $\partial\Omega$:

(2.28)
$$Z_s\partial_z\left[N(z, \bar{z})\right] + \bar{Z}_s\partial_{\bar{z}}\left[N(z, \bar{z})\right] = -\frac{i}{2}\sigma Z_{ss}\ ,\ \text{ where } N(z, \bar{z}) = f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z})$$

Integration in s gives

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -\frac{i}{2}\sigma Z_s \text{ for } z \in \partial\Omega \tag{2.29}$$

A possible integration constant on the right of (2.29) can be chosen to be zero, without loss of generality by exploiting translation freedom in choice of z, f and g. Note that since the domain Ω changes with time t, all the dependent variables and analytic functions f and g also depend on t as well, though this dependence has been suppressed until now.

It is easily checked that the normal speed of the interface is $V_n = (X_t, Y_t) \cdot (-Y_s, X_s) = \text{Im}\left[Z_t \bar{Z}_s\right]$, and $\mathbf{u} \cdot \mathbf{n} = (u, v) \cdot (-Y_s, X_s) = \text{Im}\left[(u + iv)\bar{Z}_s\right]$. So, kinematic condition $V_n = \mathbf{u} \cdot \mathbf{n}$ implies on $\partial\Omega$:

$$\text{Im}\ \left\{\left[Z_t - (u + iv)\right] \bar{Z}_s\right\} = 0 \tag{2.30}$$

The Stokes problem in 2-D reduces to determining analytic functions f, g in Ω from knowing that (2.29) is satisfied on $\partial\Omega$ at each instant of time. The boundary $\partial\Omega$ is specified initially and at each instant of time described parameterically by $z = x + iy = Z(s, t)$. The boundary evolution is described by (2.30), where velocity $\mathbf{u} = (u, v)$ is known in terms of f and g through (2.23).

This is clearly a nonlinear problem. We now discuss ways to solve it using conserved quantitites.

3. Conservation law for zero surface tension

For $\sigma = 0$, the Stress condition (2.29) reduces to

$$f(z)\ +\ z\ \bar{f}'(\bar{z})\ +\ \bar{g}'(\bar{z})\ =\ 0 \text{ on } \partial\Omega \tag{3.31}$$

Without loss of generality, we may assume $f(0) = 0$ since for any complex constant a, change of variables $f(z) \to f(z) + a$, $g'(z) \to g'(z) + \bar{a}$ has no impact on the velocity $u + iv = -f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z})$. We will now prove the following

Theorem 3.1. *Define $\zeta(z, t)$ to be a conformal map from Ω to a unit circle, mapping $z = 0$ to $\zeta = 0$. Then, as long as a smooth solution exists, arbitary analytic function M in the unit circle,*

$$\frac{d}{dt}\int_{\Omega} M(\zeta(z, t))dA = mM(0)$$

We will prove Theorem 3.1 after some preliminary lemmas and definitions.

Definition 3.1. *Let $G(x, y; t)$ be the Green's function*[(2)] *for Dirichilet BC with singularity at the origin, i.e. $G(x, y; t) + \frac{1}{2\pi}\log r$ is Harmonic in Ω and $G = 0$ on $\partial\Omega$. Define $\mathcal{G}(z, t)$ so that* $\text{Re}\left\{\mathcal{G}(z, t) + \frac{1}{2\pi}\log z\right\}$ *is analytic for $z \in \Omega$ and*

$$Re\left\{\mathcal{G}(z, t)\ +\ \frac{1}{2\pi}\ \log\ z\right\} = G(x, y; t) + \frac{1}{2\pi}\log r$$

Remark 3.2. *From Cauchy Riemann conditions, the harmonic conjugate exists in a simply connected domain and is uniquely determined up to an additive constant. Hence for given Ω, $\mathcal{G}$ is determined uniquely up to an arbitrary imaginary constant.*

---

(2) There is t dependence because $\Omega = \Omega(t)$

Lemma 3.3. *(Green's Theorem in complex form) For arbitrary C^1 function H, if $\partial\Omega$ is smooth*[3] *and traversed anti-clockwise direction,*

$$\int_\Omega \frac{\partial}{\partial \bar z} H(z,\bar z)\, dA \;=\; \frac{1}{2i} \oint_{\partial\Omega} H(z,\bar z)\, dz, \tag{3.32}$$

and

$$\int_\Omega \frac{\partial}{\partial z} H(z,\bar z)\, dA \;=\; -\frac{1}{2i} \oint_{\partial\Omega} H(z,\bar z)\, d\bar z \tag{3.33}$$

Proof. Each of the formulae simply follows from Green's theorem in the plane by separating out real and imaginary parts by noting $H = H_1 + iH_2$, $dz = dx + idy$, $d\bar z = dx - idy$. ∎

Lemma 3.4.

$$\zeta(z,t) \equiv \exp\left[-2\pi \mathcal{G}(z,t)\right] \tag{3.34}$$

conformally maps Ω into the unit circle with $z = 0$ corresponding to $\zeta = 0$

Proof. From Riemann mapping theorem (see for instance Conformal mapping by Nehari), at each t, there exists a conformal map $\zeta(z;t)$ that maps $\Omega(t)$ into the unit circle with $z = 0$ corresponding to $\zeta = 0$. The class of all such maps is of the form $e^{i\phi}\zeta$ for some constant ϕ, which may depend on t. Since $|\zeta| = 1$ on $\partial\Omega$, it follows that $-\frac{1}{2\pi}\log\left|e^{i\phi}\zeta\right| = 0$ on $\partial\Omega$ and from the analyticity of h and Taylor expansion of h at $z = 0$, and uniqueness of Green's function,

$$G(\mathbf{x};t) = -\frac{1}{2\pi}\log\left|\zeta(z;t)\right|$$

Since $\mathcal{G}(z,t)$ is defined uniquely up to an imaginary constant, it follows that we may choose the constant so that

$$\mathcal{G}(z,t) = -\frac{1}{2\pi}\log\zeta(z,t)$$

∎

Remark 3.5. *The map $\zeta(z,t)$ is also known to be smooth on $\partial\Omega$ for smooth boundaries.*

Lemma 3.6. *For evolution of drops in Stokes flow, on $\partial\Omega$,*

$$\partial_t\zeta + (u + iv)\partial_z\zeta = 0 \tag{3.35}$$

Proof. The kinematic boundary condition in (1.2) implies (see last week notes)

$$G_t \;+\; \mathbf{u}\cdot\nabla G \;=\; 0 \text{ on } \partial\Omega \tag{3.36}$$

In the complex form, since $G = \mathrm{Re}\mathcal{G}$, we can check that this implies

$$Re\left[\frac{\partial}{\partial t}\mathcal{G} \;+\; (u \;+\; i\,v)\,\mathcal{G}_z\right] \;=\; 0 \text{ on } \partial\Omega \tag{3.37}$$

Using (2.24) and (3.31),

$$Re\left[\partial_t\left(\log\frac{\zeta(z,t)}{z}\right) - 2\, f(z)\partial_z\left(\log\frac{\zeta(z,t)}{z}\right) - \;2\,\frac{f(z)}{z}\right] \;=\; 0 \text{ on } \partial\Omega \tag{3.38}$$

---

[3] This can be relaxed as you know, but we only need this stronger version here

Since $f(0) = 0$ and $z = 0$ is mapped to $\zeta = 0$, it follows that the left hand side of (3.38) is analytic inside Ω Hence, everywhere in Ω including the boundary (because of assumed smoothness)

$$\partial_t \left(\log \frac{\zeta(z,t)}{z} \right) - 2\, f(z)\, \partial_z \left(\log \frac{\zeta(z,t)}{z} \right) - 2\, \frac{f(z)}{z} = 0, \tag{3.39}$$

implying

$$\partial_t \left(\log\, \zeta\,\right) - 2\, f(z)\, \partial_z \left(\log\, \zeta\right) = 0 \tag{3.40}$$

which on using (3.31) and (2.24) implies the lemma statement. ∎

Proof of Theorem 3.1:

Define $\mathcal{M}(z,t) = M\left(\zeta(z,t)\right)$. Note that t dependence of $\mathcal{M}$ comes t dependence of conformal map $\zeta(z,t)$. As in proof of conservation of mass statements in week 1 notes, it is easily seen that

$$\frac{d}{dt} \int_\Omega \mathcal{M}(z,t)\, dA = \int_\Omega \left[\mathcal{M}_t\, (z,t) + \mathbf{u} \cdot \nabla \mathcal{M}\,\right]\, dA \tag{3.41}$$

where $\mathbf{u} = (u, v)$. However, since $\mathcal{M}(z,t) = M_1 + i M_2$ is analytic in z,

(3.42)

$$\mathbf{u} \cdot \nabla\, \mathcal{M} = u \partial_x M_1 + v \partial_y M_1 + i\left(u \partial_x M_2 + v \partial_y M_2\right) = (u+iv) \partial_x \left(M_1 + i M_2\right) = (u+iv) \mathcal{M}_z$$

Therefore (3.41) can be written as

(3.43)

$$\int_\Omega \frac{\partial}{\partial \bar z} \left[\bar z \left(\frac{\partial \mathcal{M}}{\partial t} + (u + i\, v)\, \mathcal{M}_z \right) \right] dA - \int \int_\Omega \bar z\, \frac{\partial}{\partial \bar z}\, (u + iv)\, \frac{\partial \mathcal{M}}{\partial z}\, dA$$

Using (3.32), and that $\mathcal{M}(z,t) = M(\zeta(z,t))$, where M is analytic in $\zeta = exp\left(-2\,\pi\, \mathcal{G}(z,t)\right)$, we obtain from (3.43)

(3.44)

$$\frac{1}{2i} \oint_{\partial\Omega} \bar z M'(\zeta) \left(\frac{\partial \zeta}{\partial t} + (u+iv) \zeta_z \right) dz - \int_\Omega \bar z\, \frac{\partial}{\partial \bar z} (u+iv) \frac{\partial \mathcal{M}}{\partial z}\, dA = - \int_\Omega \bar z \frac{\partial}{\partial \bar z} (u+iv) \frac{\partial M}{\partial z} dA$$

because of Lemma 3.6. Using (2.24), it follows that

$$\int_\Omega \bar z\, \left[z \bar f''(\bar z) + \bar g''(\bar z) \right]\, \mathcal{M}_z\, dA \tag{3.45}$$

can be written as

$$\int_\Omega \frac{\partial}{\partial z} \left\{ \bar z \left[z\, \bar f''(\bar z) + \bar g''(\bar z) \right] \mathcal{M} \right\} dA - \int_\Omega \frac{\partial}{\partial \bar z} \left\{ \mathcal{M} \left[\bar z \bar f'(\bar z) - \bar f(\bar z) \right] \right\} dA \tag{3.46}$$

On using Green's theorem in the form (3.32) and (3.33), it follows that expression (3.46) is equal to

(3.47)

$$-\frac{1}{2i} \oint_{\partial\Omega} \mathcal{M}\, \bar z\, \left[z\, \bar f''(\bar z) + \bar g''(\bar z) \right]\, d\bar z + \frac{1}{2i} \oint_{\partial\Omega} \mathcal{M} \left[- \bar z \bar f'(\bar z) + \bar f(\bar z) \right]\, dz$$

However, the stress condition (3.31) implies that on $\partial\Omega$,

$$\left[z\, \bar f''(\bar z) + \bar g''(\bar z) \right]\, d\bar z + \left[f'(z) + \bar f'(\bar z) \right]\, dz = 0 \tag{3.48}$$

Using this in the first integral in (3.47) and combining with the second integral, (3.47) reduces to

$$\frac{1}{2i}\oint_{\partial\Omega} \mathcal{M}\ \left[\bar{z}f'(z)\ +\ \bar{f}(\bar{z})\right]\ dz \tag{3.49}$$

Using complex conjugate of (3.31), (3.49) is reduced to

$$-\frac{1}{2i}\oint_{\partial\Omega} \mathcal{M}g'(z)\ dz\ =\ \pi\mathcal{M}(0) \tag{3.50}$$

since the source at $z = 0$ implies $g'(z) = -\frac{m}{2\pi z} + O(1)$ as $z \to 0$. It follows that

$$\frac{d}{dt}\int_{\Omega} \mathcal{M}(z,t)\ dA\ =\ \pi\mathcal{M}(0) \tag{3.51}$$

4. Reconstruction of Ω at time t

For $\zeta = \zeta(z,t)$, the conformal map defined in the last section, define moments for $k \geq 0$

$$M_k \equiv \int_{\Omega} \zeta^k dA = \frac{1}{2i}\oint_{\partial\Omega} \zeta^k \bar{z} dz$$

Using Greens theorem, it follows that

$$M_k = \frac{1}{2i}\oint_{\zeta} \bar{Z}(\zeta^{-1})Z_\zeta(\zeta)\zeta^k d\zeta,$$

where $Z(\zeta,t)$ is the inverse function of $\zeta(z,t)$ at each fixed time t.

Remark 4.1. *Using representation $Z(\zeta,t) = \sum_{n=1}^{\infty} a_n(t)\zeta^n$, it follows from contour integration that for $k \geq 0$*

$$M_k = \pi\sum_{j=1}^{\infty} j a_j \bar{a}_{j+k} \tag{4.52}$$

In the case when z is a polynomial of order N, this reduces to

$$M_k = \pi\sum_{j=1}^{N-k} j a_j \bar{a}_{j+k} \tag{4.53}$$

for $0 \leq k \leq N-1$ and $M_k = 0$ for $k \geq N$.

Theorem 4.1. *$M_k = 0$ for all $k \geq N$, iff $Z(\zeta,t) = \sum_{n=1}^{N} a_n(t)\zeta^n$. In that case, M_0, M_1, ..M_{N-1} are quadratic functions defined in (4.53).*

Proof. From Remark 4.1, a polynomial conformal map $Z(\zeta,t)$ implies $M_k = 0$ for $|k| \geq N$. Assume now that $M_k = 0$ for $k \geq N$. Extend definition so that $a_n = 0$ for $n \leq 0$ and we use (4.52) to extend M_k for $k \leq 0$, which will of course yield $M_k = \bar{M}_k$ for $k < 0$. Since $M_k = 0$ for $k \geq N$, it follows that for the extended set, $M_k = 0$ for $|k| \geq N$. Then, suppressing the t-dependence, define

$$M(\zeta) \equiv \sum_{k=-\infty}^{\infty} M_k\zeta^{-k} = \pi\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} j a_j\zeta^j \bar{a}_{j+k}\zeta^{-j-k} = \pi\zeta Z_\zeta(\zeta)\bar{Z}(1/\zeta), \tag{4.54}$$

This implies

$$\pi Z_\zeta(\zeta) = \frac{M(\zeta)}{\zeta\bar{Z}(1/\zeta)} \tag{4.55}$$

Since $M(\zeta) = \sum_{k=-N+1}^{N-1} M_k \zeta^k$, while

$$\frac{1}{\zeta \bar{Z}(1/\zeta)} = \frac{1}{a_1 + \sum_{j=1}^{\infty} \bar{a}_{j+1} \zeta^{-j}}$$

contains no positive powers of ζ in the Laurent series as $\zeta \to \infty$[4], it follows that the right side of (4.55) contains no powers of ζ larger than $N-1$ in the Laurent series at ∞. From (4.55), Z_ζ cannot contain any powers of ζ higher than $N-1$. Therefore, Z must be an N-th order polynomial. $\blacksquare$

Corollary 4.2. *If $Z(\zeta, 0) = \sum_{j=1}^{N} a_{j,0} \zeta^j$ then as long as smooth solution of for the evolution of drop exists $Z(\zeta, t) = \sum_{j=1}^{N} a_j(t) \zeta^j$ with a_j determined from*

$$M_k\left(a_1(t), a_2(t), ..a_N(t)\right) = M_k\left(a_{1,0}, ..a_{N,0}(0)\right) + tm\delta_{k,0}$$

[4] Note $a_1 \neq 0$ as otherwise Z would not be conformal at $\zeta = 0$