Week 11 Notes, Math 8610, Tanveer

1. 2-D inviscid irrotational free boundary: Conformal Mapping

Last week, we looked at the problem of bubble in a 3-D flow, found solutions of spherical bubbles and determined the motion for linear perturbation about a sphere. We did not discuss the fully nonlinear problem. There are two general approaches for nonlinear free-boundary problems in inviscid irrotational flow: Boundary integral method and conformal mapping approach. The latter is restricted to 2-D, as one might expect.

We illustrate here the formulation of the free-boundary problem for an evolving bubble in a conformal mapping representation. Further, we investigate mathematical properties of a 2-D steadily translating bubble in the absence of a body force, but in the presence of surface tension. Note that 2-D water waves may be formulated in a similar manner.

Consider the evolution of 2-D bubble of arbitrary shape where fluid is moving uniformly with speed U_0 at ∞ along the x_1-axis, say with no circulation or change of bubble volume. We introduce complex variable

$$z = x_1 + i x_2 \tag{1.1}$$

Since we have by assumption a potential flow with harmonic potential Φ in Ω outside the bubble, it is appropriate to introduce the complex potential $W(z)$ and complex velocity $\frac{dW}{dz}$:

$$W(z) = \Phi + i\Psi \ , \ \frac{dW}{dz} = \Phi_{x_1} - i\Phi_{x_2} = u_1 - i u_2 \tag{1.2}$$

So, as $z \to \infty$ the condition of uniform flow with no circulation and source becomes

$$\frac{dW}{dz} \sim U_0 + O(1/z^2) \tag{1.3}$$

Consider the conformal map

$$z = Z(\zeta, t), \tag{1.4}$$

that at each instant of time maps the domain interior of the unit circle in the ζ-plane into the exterior of the evolving bubble (See Fig. 1), with $\zeta = 0$ corresponding to $z = \infty$. This mapping must have the form

(1.5)

$$Z(\zeta, t) = -\frac{a(t)}{\zeta} + f(\zeta, t) \ , \ f(\zeta, t) = \sum_{n=0}^{\infty} a_n(t)\zeta^n \text{ convergent for } |\zeta| < 1$$

where we require $a(t) > 0$, which uniquely determines the conformal map. It is to be noted from the nature of conformal map that if we

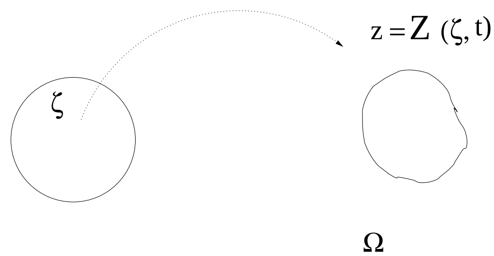

FIGURE 1. Conformal map $Z(\zeta, t)$ from ζ to z-plane

traverse the boundary $|\zeta| = 1$ counter-clockwise, then in the z-plane, the corresponding image under $z = Z(\zeta, t)$ traverses the boundary of the bubble clockwise. It is convenient to define

$$w(\zeta, t) = W(Z(\zeta, t), t) \tag{1.6}$$

We notice that through chain-rule.

$$\frac{dW}{dz}\frac{dZ}{d\zeta} = \frac{dw}{d\zeta} \tag{1.7}$$

$$\partial_t W + \frac{dW}{dz}\partial_t Z = \partial_t w \ , \ \text{hence } \partial_t W = \partial_t w - \frac{\frac{dw}{d\zeta}}{\frac{dZ}{d\zeta}}\partial_t Z \tag{1.8}$$

The free boundary $\partial\Omega$ is characterized in the ζ-plane by $|\zeta| = 1$; we use the paramterization $\zeta = e^{i\nu}$ on the free boundary and so, it is parametrically determined by $z = Z(e^{i\nu}, t)$ We denote by θ, the angle made by the unit tangent with the Re z-axis, as the the bubble boundary is traversed *clockwise* and s the arclength along the bubble boundary. Then curvature of the bubble

$$\kappa = -\frac{d\theta}{ds} = -[\frac{ds}{d\nu}]^{-1}\frac{d}{d\nu}\arg\frac{dZ}{d\nu} = -\frac{1}{|\frac{dZ}{d\nu}|}\text{Im}\left\{\frac{d}{d\nu}\log Z_\nu\right\} \tag{1.9}$$

Recall pressure condition on the free boundary $\partial\Omega$:

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + V = \frac{\sigma}{\rho}\kappa \tag{1.10}$$

Because of relations (1.7)-(1.9), we obtain the following pressure condition on $\zeta = e^{i\nu}$, on using $i\zeta\frac{dZ}{d\zeta} = \frac{dZ}{d\nu}$:

$$\text{Re}\left\{\partial_t w - \frac{\frac{dw}{d\zeta}}{\frac{dZ}{d\zeta}}\partial_t Z\right\} + \frac{1}{2}\left|\frac{\frac{dw}{d\zeta}}{\frac{dZ}{d\zeta}}\right|^2 + V = -\frac{\sigma}{\rho|\frac{dZ}{d\zeta}|}\text{Re}\left\{\zeta\frac{d}{d\zeta}\log[\zeta Z_\zeta]\right\} \tag{1.11}$$

If body force potential is due to gravity, *i.e.* $V = gx_2$, we write it as $g\ \mathrm{Im}Z$. We now derive the kinematic boundary condition. Since the free boundary is given by $|\zeta| = 1$, or $\log|\zeta(z,t)| = 0$, it follows that

$$(1.12)\quad \partial_t \log|\zeta(z,t)| + [u_1\partial_{x_1} + u_2\partial_{x_2}] \log|\zeta(z,t)| = 0$$
$$= \mathrm{Re}\left\{\partial_t \log\zeta(z,t) + \left(\frac{dW}{dz}\right)^* \frac{d}{dz}\log\zeta(z,t)\right\}$$

We note that

$$(1.13)\qquad \frac{d\zeta}{dz} = \frac{1}{Z_\zeta},\ \text{ while } 0 = \partial_t z = \partial_t[Z(\zeta(z,t),t)] = \partial_t Z + \frac{dZ}{d\zeta}\zeta_t$$

Hence from (1.12), we obtain the kinematic condition to be equivalent to

$$(1.14)\qquad \mathrm{Re}\left\{-\frac{\partial_t Z}{\zeta\frac{dZ}{d\zeta}} + \frac{1}{\zeta Z_\zeta}\left(\frac{\frac{dw}{d\zeta}}{\frac{dZ}{d\zeta}}\right)^*\right\} = 0$$

or by using the fact that on $\zeta = e^{i\nu}$,

$$\mathrm{Re}\left\{\frac{1}{\zeta}\left(\frac{dw}{d\zeta}\right)^*\right\} = \mathrm{Re}\left\{\zeta\frac{dw}{d\zeta}\right\},$$

$$(1.15)\qquad \mathrm{Re}\left\{\frac{Z_t}{\zeta Z_\zeta} - \frac{\zeta\frac{dw}{d\zeta}}{|Z_\zeta|^2}\right\} = 0$$

Equation (1.15) is the kinematic condition for any potential 2-D free boundary problem; not just for a bubble. At $z = \infty$, which corresponds to $\zeta = 0$, the asymptotic condition (1.3) becomes equivalent to requiring that as $\zeta \to 0$,

$$(1.16)\qquad w \sim -\frac{U_0 a(t)}{\zeta} + O(1)$$

It is clear that, since $w(\zeta,t)$ must be regular at other points in $|\zeta| < 1$, w has the representation:

(1.17)

$$w = -\frac{U_0 a(t)}{\zeta} + \sum_{n=0}^{\infty} b_n(t)\zeta^n\ ,\ \text{ where series convergent for } \ |\zeta| < 1$$

For the evolution of the free surface, we have to solve (1.11) and (1.15) for w and Z, each of which have the representation (1.5) and (1.17). It is not difficult to show that if the boundary is smooth, the convergence of the series representation extends to $|\zeta| = 1$. Indeed, it the boundaries are analytic, the convergence extends slightly beyond the unit circle.

Generally, because of the complication of the equations (1.11) and (1.15), these problems cannot be solved exactly. However, numerically, this form of representation is useful since we can truncate the series to (1.5) and (1.17) to a finite number of terms and solve the resulting ODEs for $a, \{a_n\}, \{b_n\}$ and hence for the shape of the free boundary. The advantage of the conformal mapping representation is that we have converted a free boundary problem into a fixed boundary problem, the domain now being $|\zeta| < 1$. In the next section, we will consider the special case of a steady translating bubble with no body force, *i.e.* $V = 0$, where further progress can be made through analysis.

2. Steadily translating 2-D inviscid irrotational bubble:

In this case there is no-time dependence, and we are neglecting gravity or any other body force, *i.e.* $V = 0$. Then, from (1.15), we have $\mathrm{Re}\ \zeta \frac{dw}{d\zeta} = 0$, implying $\frac{d}{d\nu}\mathrm{Im}\ w = 0$, which without loss of generality means

$$\Psi = \mathrm{Im}\ w = 0 \text{ on } \ \zeta = e^{i\nu} \tag{2.18}$$

From (1.16), it follows that without any loss of generality,

$$w = -U_0 a \left(\frac{1}{\zeta} + \zeta\right) = -U_0 a \omega(\zeta) \tag{2.19}$$

The pressure boundary condition reduces to

$$\frac{1}{2} U_0^2 a^2 \left|\frac{\frac{d\omega}{d\zeta}}{\frac{dZ}{d\zeta}}\right|^2 = C - \frac{\sigma}{\rho |\frac{dZ}{d\zeta}|} \mathrm{Re} \left\{ \zeta \frac{d}{d\zeta} \log[\zeta Z_\zeta] \right\} \tag{2.20}$$

We have a constant C on the right because steady flow only implies that $\mathrm{Re}[\partial_t w] = -C$ is independent of time, though not necessarily zero. It is convenient to rescale Z so that

$$Z(\zeta) = a f(\zeta) \tag{2.21}$$

Then, (2.20) reduces to one parameter equation on $|\zeta| = 1$:

$$b \left|\frac{\omega_\zeta}{f_\zeta}\right|^2 = \gamma - \frac{1}{|f_\zeta|} \mathrm{Re} \left\{ 1 + \zeta \frac{f_{\zeta\zeta}}{f_\zeta} \right\} \ , \text{ where } b = \frac{a \rho U_0^2}{2\sigma} \tag{2.22}$$

It is convenient to introduce

$$y(\zeta) = f_\zeta^{1/2}(\zeta) \tag{2.23}$$

The function $y(\zeta)$ will then have a convergent series representation

$$y(\zeta) \ = \ \frac{1}{\zeta} + \sum_{n=0}^{\infty} c_n \ \zeta^n \tag{2.24}$$

for $|\zeta| \le 1$, where it will be nonzero as well since $f_\zeta \neq 0$ for $|\zeta| \le 1$ for a smooth boundary. The pressure condition (2.22) can be rewritten as

$$|y|^2 \left[1 + 2Re\left(\zeta \frac{y_\zeta}{y}\right)\right] + b\,|\zeta^2 - 1|^2 - \gamma\,|y|^4 = 0, \tag{2.25}$$

2.1. **General properties of the analytically continued $y(\zeta)$.** With a view to understanding the analytic properties of the conformal mapping function $Z(\zeta)$, we investigate the related function $y(\zeta)$ outside the unit disk. In the process, it becomes necessary to analytically continue the Bernoulli equation (2.25) off the boundary $|\zeta| = 1$. Equation (2.25) is not suitable for such a continuation since it involves absolute values. However, we notice that on $|\zeta| = 1$, $\zeta^* = 1/\zeta$, $[y(\zeta)]^* = \bar{y}(1/\zeta)$ and $y_\zeta^* = \bar{y}_\zeta(1/\zeta)$, where the superscript * denotes complex conjugate and $\bar{y}(\zeta)$ is defined by

$$\bar{y}(\zeta) = 1/\zeta + \sum_{n=0}^{\infty} c_n^* \zeta^n, \tag{2.26}$$

Using these properties, it is found that (2.25) is equivalent to

$$y_\zeta - q_1 y - q_2 - \gamma q_3 y^2 = 0 \tag{2.27}$$

on $|\zeta| = 1$, where

$$q_1 = -\frac{1}{\zeta}\left[1 + \frac{1}{\zeta}\,\frac{\bar{y}_\zeta(1/\zeta)}{\bar{y}(1/\zeta)}\right], \tag{2.28}$$

$$q_2 = \frac{b(\zeta^2 - 1)^2}{\zeta^3 \bar{y}(1/\zeta)}, \tag{2.29}$$

$$q_3 = \frac{1}{\zeta}\,\bar{y}(1/\zeta), \tag{2.30}$$

Note that with the conditions on y on $|\zeta| = 1$, each of q_1, q_2, q_3, y and therefore the left hand side of (2.27) is analytic on $|\zeta| = 1$. It follows from the well known principle of analytic continuation that (2.27) holds everywhere in the complex ζ plane. Singularities of the conformal mapping function Z and hence of y, aside from the pole at $\zeta = 0$, can occur in $|\zeta| > 1$ only if a solution to (2.27) encounters singularities.

Also from the property that y is analytic and nonzero inside and on the unit circle except for a simple pole at $\zeta = 0$, it follows that each of q_1, q_2 and q_3 defined in (2.28)-(2.30) will be analytic everywhere in $|\zeta| \ge 1$, with the following general behavior at ∞

$$q_1(\zeta) = O(\zeta^{-2}), \tag{2.31}$$

$$q_2 \sim b + O(1/\zeta), \tag{2.32}$$

$$q_3 \sim 1 + O(1/\zeta). \tag{2.33}$$

Note, also that since y is nonzero inside the unit circle, q_3 defined by (2.30) cannot be zero anywhere outside the unit ζ circle. This property will be important later.

The simplest case for which there is a trivial exact solution is when $\gamma = -1$ corresponding to which $b = 0$ and

$$y(\zeta) = \frac{1}{\zeta} \tag{2.34}$$

This is for a circular bubble that is stationary is a quiescent fluid. This solution is approached when surface tension effects are very strong compared to inertial effects.

The other case for which there is a known exact solution is when $\gamma = 0$. The y^4 term drops out of (2.27) and the a priori knowledge of analyticity of q_1 and q_2 implies that any solution to this equation has the well known properties of a solution to a linear first order differential equation with analytic coefficients in $|\zeta| \geq 1$. This is true despite the fact that q_1 and q_2 depend on y. As a consequence of the analyticity of q_1 and q_2, it follows that y cannot have any singularity in the finite ζ plane outside the unit circle. Further, since y is related to the conformal mapping function $z(\zeta, t)$ through (2.23), the only singularity of y inside the unit circle can be a simple pole at $\zeta = 0$. Examining the neighborhood of $\zeta = \infty$, it is easy to see from (2.27) and properties (2.31)-(2.33) that $\zeta = \infty$ is a regular singular point. From a Froebinius series representation for y in the variable in $1/\zeta$, we conclude from (2.27) that $y \sim b\,\zeta + O(1)$, as $\zeta \to \infty$. Combining all the information about y, we conclude that

$$y = 1/\zeta + a_0 + b\,\zeta, \tag{2.35}$$

for some complex a_0. Substituting this form into (2.27), with q_1 and q_2 determined accordingly, the ζ^{-1} term of the Laurent series expansion of the left hand side of (2.27) about $\zeta = \infty$, when equated to zero, results in

$$a_0 = 0.$$

Similarly, from the constant term of this Laurent series, we obtain the condition

$$b^{-1}\,(b+1)(1-3b) = 0. \tag{2.36}$$

Equation (2.36) implies that $b = \frac{1}{3}$, since it must be a finite nonnegative number. Thus, for $\gamma = 0$,

$$Z_\zeta = ay^2 = a(1/\zeta + \zeta/3)^2. \tag{2.37}$$

More generally, for $\gamma \neq 0$, it is possible to recognize that (2.27) has the form of a Ricatti equation, and therefore can be related to a linear second order equation with a transformation. It is possible to conclude that $b = b(\gamma)$ and

$$y = \frac{1}{\zeta} + \sum_{j=1}^{\infty} \frac{2r_j\zeta}{\zeta^2 - \zeta_j^2}\ , \text{ implying } Z_\zeta = a\left\{\frac{1}{\zeta} + \sum_{j=1}^{\infty} \frac{2r_j\zeta}{\zeta^2 - \zeta_j^2}\right\}^2, \tag{2.38}$$

where ζ_j is a discrete sets with the following asymptotic behavior as $j \to +\infty$:

$$\zeta_j \sim \gamma^{-1/2} b^{-1/2}\left[(2j-1)\frac{\pi}{2} + \phi\right], \tag{2.39}$$

$$r_j \sim \frac{1}{\gamma} \tag{2.40}$$