More Week 12 Notes, Math 8610, Tanveer

1. Steady Stokes Equation – preliminaries

We now discuss the steady Stokes equation. The objective here is to build the necessary tools needed to prove that steady Stokes equation has a unique solution; as a bi-product, we are able able to bridge some of the gaps in steady Navier-Stokes analysis.

Recall the domain $\Omega \subset \mathbb{R}^n$ is an open domain on one side of Γ, which is Lipschitz. We will denote $\{\mathcal{O}_j\}_{j\in J}$ as an open cover for Γ.

Definition 1.1. *Let $\mathcal{D}(\Omega)$ and $\mathcal{D}\left(\overline{\Omega}\right)$ be the space of C^∞ functions with compact support contained Ω and $\overline{\Omega}$ respectively.*

Definition 1.2. *Let $\mathcal{V}$ be the space of functions*

$$\mathcal{V} = \{u \in \mathcal{D}(\Omega) \ , \nabla \cdot u = 0\} \tag{1.1}$$

We define the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $H_0^1(\Omega)$ to be H and V respectively.

1.1. The space $E(\Omega)$.

Definition 1.3. *We define the auxiliary space $E(\Omega)$*

$$E(\Omega) = \left\{u \in L^2(\Omega) \ , \nabla \cdot u \in L^2(\Omega)\right\} \tag{1.2}$$

This is a Hilbert space with scalar product

$$(u, v)_{E(\Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v) \tag{1.3}$$

and norm $\|u\|_{E(\Omega)} = \left[(u, u)_{E(\Omega)}\right]^{1/2}$.

Theorem 1.1. *Let Ω be a Lipschitz open set in $\mathbb{R}^n$. Then $\mathcal{D}\left(\overline{\Omega}\right)$ is dense in $E(\Omega)$.*

Proof. Let $u \in E(\Omega)$. We see to show u is th limit of functions in $\mathcal{D}\left(\overline{\Omega}\right)$. We will assume without any loss of generality that u has bounded support even when Ω is unbounded; otherwise we replace u may be replaced by $u\phi(x/a)$ where $\phi(x)$ is a smooth positive cutoff function that is zero for $|x| \geq 2$ and equal to 1 for $|x| \leq 1$ and it is known that $u(x)\phi(x/a) \to u$ in $L^2(\Omega)$ and $\nabla \cdot (u(x)\phi(x/a)) \to \nabla \cdot u$ as $a \to \infty$.
i. First, consider the case $\Omega = \mathbb{R}^n$. Then we can introduce a standard mollification $u_\epsilon \equiv \rho_\epsilon * u \in \mathcal{D}\left(\overline{\Omega}\right)$ and it is known that $u_\epsilon \to u$ and $\nabla \cdot u_\epsilon = \rho_\epsilon * (\nabla \cdot u) \to \nabla \cdot u$, each in $L^2(\mathbb{R}^n)$ and the theorem is proved.
ii. For the general case when $\Omega \neq \mathbb{R}^n$, we consider the open cover $\{\mathcal{O}_j\}_{j\in J}$ of Γ. It is clear that the sets Ω, $\{\mathcal{O}_j\}_{j\in J}$ is an open cover of

$\overline{\Omega}$ and we consider a partition of unity so that for any $x \in \overline{\Omega}$,

$$1 = \phi(x) + \sum_{j \in J} \phi_j(x) \text{ , where } \phi \in \mathcal{D}(\Omega) \text{ , } \phi_j \in \mathcal{D}(\mathcal{O}_j) \tag{1.4}$$

So, if we decompose

$$u = \phi u + \sum_{j \in J} \phi_j u \tag{1.5}$$

Since the function ϕu has compact support in Ω, the argument in **i** may be repeated to show that $\rho_\epsilon * (\phi u) \in \mathcal{D}(\Omega)$ and approaches ϕu in $E(\Omega)$ norm.

Let us now consider one of the $u_j = \phi_j u$ that is not identically 0. Since the set $\mathcal{O}'_j = \mathcal{O}_j \cap \Omega$ is star shaped with respect to one of its points, we will translate that point, w.l.o.g to zero. We define for $\lambda > 0$, the linear scaling transformation σ_λ: $x \to \lambda x$ and $\sigma_\lambda \circ v$ to be the function $x \to v(\lambda x)$. Since $\mathcal{O}'_j$ is star shaped and Lipschitz with respect to 0, it follows that

$$\mathcal{O}'_j \subset \overline{\mathcal{O}'_j} \subset \sigma_\lambda \mathcal{O}'_j \text{ for } \lambda > 1 \text{ , } \sigma_\lambda \mathcal{O}'_j \subset \overline{\sigma_\lambda \mathcal{O}'_j} \subset \mathcal{O}'_j \text{ for } 0 < \lambda < 1 \tag{1.6}$$

Now, we claim the restriction of $\sigma_{1/\lambda} \circ u_j$ to $\mathcal{O}'_j$ for $\lambda > 1$ converges to u_j in $E\left(\mathcal{O}'_j\right)$ (or $E(\mathcal{O})$) as $\lambda \to 1$. We note that

$$\int_{x \in \sigma_\lambda \mathcal{O}'_j} \left| u_j\left(\lambda^{-1} x\right) \right|^2 dx = \lambda^n \int_{y \in \mathcal{O}'_j} \left| u_j(y) \right|^2 dy \tag{1.7}$$

Therefore, it is enough to show that $\sigma_{1/\lambda} \circ u$ restricted to $\mathcal{O}'_j$ converges to u. But this is true for $u \in \mathcal{D}(\mathcal{O}'_|)$, which is dense subset of $L^2(\mathcal{O}'_j)$.

Furthermore, if $\psi_j \in \mathcal{D}\left(\sigma_\lambda \mathcal{O}'_j\right)$ with $\psi_j = 1$ in $\mathcal{O}'_j$ then $w_j = \psi_j \sigma_{1/\lambda} \circ u_j \in E(\mathbb{R}^n)$ and has compact support in $\sigma_\lambda \mathcal{O}'_j \subset \mathbb{R}^n$. Repeating the argument in part **i**, there exists a sequence of functions in $\mathcal{D}\left(\sigma_\lambda \mathcal{O}'_j\right)$ which converge to w_j in $E\left(\sigma_\lambda \mathcal{O}'_j\right)$ and therefore to u_j in $E(\mathcal{O}'_j)$ as $\lambda \to 1$. ∎

Now, we seek to prove a trace theorem when Ω is an open bounded set with a C^2 boundary Γ. We seek to define normal velocity component $u \cdot n$ on Γ for $u \in E(\Omega)$, even when it is not defined pointwise.

Remark 1.4. First we state a few PDE results without proof for such a domain. It is known that there exists a linear continuous operator, usually called the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$, which agrees with point values u on the boundary if $u \in C^2\left(\overline{\Omega}\right)$. The image $\gamma_0\left(H^1(\Omega)\right)$ is a dense subspace of $L^2(\Gamma)$ and denoted by $H^{1/2}(\Gamma)$. More over, there is a linear contious operator $l_\Omega : H^{1/2}(\Gamma) \to H^1(\Omega)$ such that $\gamma_0 \circ l_0 = I$ (identity) on Γ.

Definition 1.5. *Define* $H^{-1/2}(\Gamma)$ *is the dual space of* $H^{1/2}(\Gamma)$

Remark 1.6. Note that

$$H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma) \tag{1.8}$$

Theorem 1.2. *(Trace Theorem in* $E(\Omega)$*): Let* Ω *be an open bounded set of class* $\mathcal{C}^2$*. Then there exists a continous linear operator* $\gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma)$ *such that* $\gamma_\nu u = u \cdot n$ *on* Γ *for* $u \in \mathcal{D}\left(\overline{\Omega}\right)$*. Futher, for any* $u \in E(\Omega)$ *and scalar function* $w \in H^1(\Omega)$

$$(u, \nabla w) + (\nabla \cdot u, w) = (\gamma_\nu u, \gamma_0 w) \tag{1.9}$$

Proof. Let $\phi \in H^{1/2}(\Gamma)$ and $w \in H^1(\Omega)$ such that $\gamma_0 w = \phi$. For $u \in E(\Omega)$, define corresponding functional

$$X_u(\phi) = (\nabla \cdot u, w) + (u, \nabla w) = \int_\Omega (w \nabla \cdot u + u \cdot \nabla w)\, dx \tag{1.10}$$

First, we note that $X_u(\phi)$ is independent of the particular choice of w since if we had $\gamma_0 w_1 = \gamma_0 w_2 = \phi$, then the difference of two calculations of $X_u(\phi)$ involving w_1 and w_2 respectively would be

$$(\nabla \cdot u, w) + (u, \nabla w)\ , \text{where } w = w_1 - w_2 \tag{1.11}$$

Since $\gamma_0 w = \gamma_0 w_1 - \gamma_0 w_1 = \phi - \phi = 0$, we know that $w \in H_0^1(\Omega)$ and can be approximated by a sequence $w_m \in \mathcal{D}(\Omega)$ and integration by parts gives

$$(\nabla \cdot u, w_m) + (u, \nabla w_m) = 0 \tag{1.12}$$

Therefore, $X_u(\phi)$ does not depend on the particular choice of $w \in H^1(\Omega)$. We define one such $w = l_\Omega \phi$. Then, from Cauchy-Schwartz,

$$\left| X_u(\phi) \right| \leq \|u\|_{E(\Omega)} \|w\|_{H^1(\Omega)} \leq c_0 \|u\|_{E(\Omega)} \|\phi\|_{H^{1/2}(\Gamma)}, \tag{1.13}$$

where c_0 is the norm of the bounded linear operator l_Ω. Therefore the mapping $\phi \to X_u(\phi)$ is a linear continuous mapping from $H^{1/2}(\Gamma)$ to $\mathbb{R}$. Thus, there exists $g = g(u) \in H^{-1/2}(\Gamma)$ such that $X_u(\phi) = \langle g, \phi \rangle$. Further, it is clear that

$$\|g\|_{H^{-1/2}(\Gamma)} \leq c_0 \|u\|_{E(\Omega)} \tag{1.14}$$

We define $\gamma_\nu u = g(u)$, which is a continuous linear mapping from $E(\Omega)$ into $H^{-1/2}(\Gamma)$. Furthermore, for $u, w \in \mathcal{D}(\overline{\Omega})$, then, using divergence theorem,

$$X_u(\phi) = \int_\Omega \nabla \cdot (wu)\, dx == \int_\Gamma \phi u \cdot n dx = \langle u \cdot n, \gamma_0 w \rangle \tag{1.15}$$

Since for these functions w, $\gamma_0 w$ form a dense set in $H^{1/2}(\Gamma)$, it follows that the formula $X_u(\phi) = \langle u \cdot n, \phi \rangle$ is true for any $\phi \in H^{1/2}(\Gamma)$ and therefore, $\gamma_\nu u = u \cdot n$ for $u \in \mathcal{D}(\Omega)$. ∎

Remark 1.7. The operator γ_ν actually maps $E(\Omega)$ **onto** $H^{-1/2}(\Gamma)$. To show this let $\phi \in H^{-1/2}(\Gamma)$ such that $\langle \phi, 1 \rangle = 0$. Then the Neumann problem

$$\Delta p = 0 \ , \text{in } \Omega \ \frac{\partial p}{\partial n} = \phi \ , \text{on } \Gamma \tag{1.16}$$

has a weak solution $p = p(\phi) \in H^1(\Omega)$, which is unique up to an additive constant. For one of these solutions define $u = \nabla p$. It is clear that $u \in E(\Omega)$ and $\gamma_\nu u = \frac{\partial p}{\partial n} = \phi$. Further, we can easily construct $u_0 \in C^1(\overline{\Omega})$ with $\gamma_\nu u_0 = 1$. Then for any $\psi \in H^{1/2}(\Gamma)$, decomposing

$$\psi = \phi + \frac{\langle \psi, 1 \rangle}{\text{measure}\Gamma}, \tag{1.17}$$

we can define u depending on ψ with $\gamma_\nu u = \psi$ simply by setting

$$u = \nabla p + \frac{\langle \psi, 1 \rangle}{\text{measure of } \Gamma} \mathbf{u}_0, \tag{1.18}$$

where p is determine from (1.16) in terms of ϕ. Moreever the mapping $\psi \to u(\psi)$ is a linear continuous mapping from $H^{-1/2}(\Gamma)$ into $E(\Omega)$, *i.e.* is a lifting operator l_Ω.

Definition 1.8. *Define $E_0(\Omega)$ to be the closure of $\mathcal{D}(\Omega)$ in $E(\Omega)$.*

We have the following theorem

Theorem 1.3. *The kernel of γ_ν is equal to $E_0(\Omega)$.*

Proof. If $u \in E_0(\Omega)$, then by definition, there exists a sequence $u_m \in \mathcal{D}(\Omega)$ which converges to u in $E(\Omega)$. Since $u_m = 0$ on boundary Γ, it follows that $\gamma_\nu u_m = 0$ and therefore $\gamma_\nu u = 0$.

Now assume for $u \in E(\Omega)$, $\gamma_\nu u = 0$. Since $u \in E(\Omega)$, it can be approximated arbitrarily closely by functions in $\mathcal{D}(\overline{\Omega})$. Let $\Phi \in \mathcal{D}(\mathbb{R}^n)$ and ϕ be the restriction of Φ to Ω. Since $\gamma_\nu u = 0$, it follows that $\langle \gamma_\nu u, \gamma_0 \phi \rangle = 0$, implying

$$\int_\Omega (\phi \nabla \cdot u + u \cdot \nabla \phi) \, dx = 0 \tag{1.19}$$

Define for functions v in Ω extensions to $\mathbb{R}^n$ so that $\overline{v}(x) = v(x)$ for $x \in \Omega$ and $\overline{v}(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then, we may re-write relation

(1.19) as

(1.20)

$$\int_{\mathbb{R}^n} \left(\Phi\overline{\nabla \cdot u} + \overline{u} \cdot \nabla\Phi\right) dx = 0 = \int_{\mathbb{R}^n} \left(\Phi\nabla \cdot \overline{u} + \overline{u} \cdot \nabla\Phi\right) dx \text{ , for any } \Phi \in \mathcal{D}(\mathbb{R}^n)$$

This implies $\overline{\nabla \cdot u} = \nabla \cdot \overline{u}$ and therefore $\overline{u} \in E(\mathbb{R}^n)$.

Now carry out the same steps as in Theorem 1.1. We use as before an open covering of $\overline{\Omega}$ using $\{\mathcal{O}_j\}_{j\in J}$ and Ω and using partition of unity, we reduce the general case to one where u has only support in $\mathcal{O}_j \cap \overline{\Omega}$. Then for $0 < \lambda < 1$, $\sigma_\lambda \circ \overline{u}$ has support within $\mathcal{O}'_j$ and yet converges to $\overline{u}$ in $E(\mathbb{R}^n)$ as $\lambda \to 1$. On the otherhand, using standard molification, one can approximate $\sigma_\lambda \circ \overline{u}$ by functions in $\mathcal{D}(\Omega)$. ∎

Remark 1.9. If the set Ω is unbounded or if its boundary is not smooth, some partial results remain true: for example, if $u \in E(\Omega)$, we can define $\gamma_\nu u$ on each bounded part Γ_0 of Γ of class C^2, and $\gamma_\nu u \in H^{-1/2}(\Gamma_0)$. If Ω is smooth but unbounded or if the boundary is the union of a finite number of bounded $(n-1)$-dimensional manifolds of class C^2, then $\gamma_\nu u$ is defined in this way on all Γ. Nevertheless, the generalized Stokes formula (1.9) does not hold.

Remark 1.10. If $u \in H^1(\Omega)$, then we will assume the following results as known: (a) There exists continuous linear mapping $\gamma_0 : \mathcal{H}^1(\Omega) \to L^2(\Gamma)$ such that $\gamma_0 u = u\Big|_\Gamma$ for every $u \in \mathcal{D}(\overline{\Omega})$. We denote $\mathcal{H}^{1/2}(\Gamma) = \gamma_0(H^1(\Omega))$; (b) There exists a lifting operator $l_\Omega : \mathcal{H}^{1/2}(\Gamma) \to H^1(\Omega)$ such that $\gamma_0 \circ l_\Omega = I$, the identity operator on $\mathcal{H}^{1/2}(\Gamma)$; $\mathcal{H}^{1/2}(\Gamma_0)$ is equipped with the norm carried by γ_0. (c) Ω is a Lipschitz set. Then, all the preceding results can be extended to this case. Theorems 1.1-1.3 hold. The proof of Theorem 1.2 leads to a definition of $\gamma_\nu u \in \mathcal{H}^{-1/2}(\Gamma)$, which is the dual space of $\mathcal{H}^{1/2}(\Gamma)$. The generalized Stokes formula (1.9) is also valid.