1. **Steady Stokes Equation — preliminaries**

We now discuss the steady Stokes equation. The objective here is to build the necessary tools needed to prove that steady Stokes equation has a unique solution; as a by-product, we are able to bridge some of the gaps in steady Navier-Stokes analysis.

Recall the domain $\Omega \subset \mathbb{R}^n$ is an open domain on one side of $\Gamma$, which is Lipschitz. We will denote $\{O_j\}_{j \in J}$ as an open cover for $\Gamma$.

**Definition 1.1.** Let $\mathcal{D}(\Omega)$ and $\mathcal{D}(\overline{\Omega})$ be the space of $C^\infty$ functions with compact support contained $\Omega$ and $\overline{\Omega}$ respectively.

**Definition 1.2.** Let $V$ be the space of functions

$$V = \{ u \in \mathcal{D}(\Omega), \nabla \cdot u = 0 \}$$

We define the closure of $V$ in $L^2(\Omega)$ and $H^1_0(\Omega)$ to be $H$ and $V$ respectively.

1.1. **The space $E(\Omega)$.**

**Definition 1.3.** We define the auxiliary space $E(\Omega)$

$$E(\Omega) = \{ u \in L^2(\Omega), \nabla \cdot u \in L^2(\Omega) \}$$

This is a Hilbert space with scalar product

$$(u, v)_{E(\Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v)$$

and norm $\|u\|_{E(\Omega)} = \left[ (u, u)_{E(\Omega)} \right]^{1/2}$.

**Theorem 1.1.** Let $\Omega$ be a Lipschitz open set in $\mathbb{R}^n$. Then $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Omega)$.

**Proof.** Let $u \in E(\Omega)$. We see to show $u$ is the limit of functions in $\mathcal{D}(\overline{\Omega})$. We will assume without any loss of generality that $u$ has bounded support even when $\Omega$ is unbounded; otherwise we replace $u$ may be replaced by $u\phi(x/a)$ where $\phi(x)$ is a smooth positive cutoff function that is zero for $|x| \geq 2$ and equal to 1 for $|x| \leq 1$ and it is known that $u(x)\phi(x/a) \to u$ in $L^2(\Omega)$ and $\nabla \cdot (u(x)\phi(x/a)) \to \nabla \cdot u$ as $a \to \infty$.

i. First, consider the case $\Omega = \mathbb{R}^n$. Then we can introduce a standard mollification $u_\epsilon \equiv \rho_\epsilon * u \in \mathcal{D}(\overline{\Omega})$ and it is known that $u_\epsilon \to u$ and $\nabla \cdot u_\epsilon = \rho_\epsilon * (\nabla \cdot u) \to \nabla \cdot u$, each in $L^2(\mathbb{R}^n)$ and the theorem is proved.

ii. For the general case when $\Omega \neq \mathbb{R}^n$, we consider the open cover $\{O_j\}_{j \in J}$ of $\Gamma$. It is clear that the sets $\Omega_j, \{O_j\}_{j \in J}$ is an open cover of
and we consider a partition of unity so that for any \( x \in \Omega \),
\[
1 = \phi(x) + \sum_{j \in J} \phi_j(x), \text{ where } \phi \in \mathcal{D}(\Omega), \phi_j \in \mathcal{D}(\mathcal{O}_j)
\]
(1.4)
So, if we decompose
\[
u = \phi \nu + \sum_{j \in J} \phi_j \nu
\]
(1.5)
Since the function \( \phi \nu \) has compact support in \( \Omega \), the argument in i may be repeated to show that \( \rho \ast (\phi \nu) \in \mathcal{D}(\Omega) \) and approaches \( \phi \nu \) in \( E(\Omega) \) norm.

Let us now consider one of the \( u_j = \phi_j \nu \) that is not identically 0. Since the set \( \mathcal{O}_j' = \mathcal{O}_j \cap \Omega \) is star shaped with respect to one of its points, we will translate that point, w.l.o.g to zero. We define for \( \lambda > 0 \), the linear scaling transformation \( \sigma_{\lambda}: x \rightarrow \lambda x \) and \( \sigma_{\lambda} \circ v \) to be the function \( x \rightarrow v(\lambda x) \). Since \( \mathcal{O}_j' \) is star shaped and Lipschitz with respect to 0, it follows that
\[
\mathcal{O}_j' \subset \overline{\mathcal{O}_j'} \subset \sigma_{\lambda} \mathcal{O}_j' \subset \mathcal{O}_j' \text{ for } 0 < \lambda < 1
\]
Now, we claim the restriction of \( \sigma_{1/\lambda} \circ u_j \) to \( \mathcal{O}_j' \) for \( \lambda > 1 \) converges to \( u_j \) in \( E(\mathcal{O}_j') \) (or \( E(\mathcal{O}) \)) as \( \lambda \to 1 \). We note that
\[
\int_{x \in \sigma_{\lambda} \mathcal{O}_j'} \left| u_j(\lambda^{-1} x) \right|^2 dx = \lambda^n \int_{y \in \mathcal{O}_j'} \left| u_j(y) \right|^2 dy
\]
(1.7)
Therefore, it is enough to show that \( \sigma_{1/\lambda} \circ u \) restricted to \( \mathcal{O}_j' \) converges to \( u \). But this is true for \( u \in \mathcal{D}(\mathcal{O}_j') \), which is dense subset of \( L^2(\mathcal{O}_j') \).

Furthermore, if \( \psi_j \in \mathcal{D}(\sigma_{\lambda} \mathcal{O}_j') \) with \( \psi_j = 1 \) in \( \mathcal{O}_j' \) then \( w_j = \psi_j \sigma_{1/\lambda} \circ u_j \in E(\mathbb{R}^n) \) and has compact support in \( \sigma_{\lambda} \mathcal{O}_j' \subset \mathbb{R}^n \). Repeating the argument in part i, there exists a sequence of functions in \( \mathcal{D}(\sigma_{\lambda} \mathcal{O}_j') \) which converge to \( w_j \) in \( E(\mathcal{O}_j') \) and therefore to \( u_j \) in \( E(\mathcal{O}_j') \) as \( \lambda \to 1 \). \( \blacksquare \)

Now, we seek to prove a trace theorem when \( \Omega \) is an open bounded set with a \( C^2 \) boundary \( \Gamma \). We seek to define normal velocity component \( u \cdot n \) on \( \Gamma \) for \( u \in E(\Omega) \), even when it is not defined pointwise.

Remark 1.4. First we state a few PDE results without proof for such a domain. It is known that there exists a linear continuous operator, usually called the trace operator \( \gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma) \), which agrees with point values \( u \) on the boundary if \( u \in C^2(\overline{\Omega}) \). The image \( \gamma_0(H^1(\Omega)) \) is a dense subspace of \( L^2(\Gamma) \) and denoted by \( H^{1/2}(\Gamma) \). More over, there is
a linear continuous operator \( l_\Omega : H^{1/2}(\Gamma) \to H^1(\Omega) \) such that \( \gamma_0 \circ l_0 = I \) (identity) on \( \Gamma \).

**Definition 1.5.** Define \( H^{-1/2}(\Gamma) \) is the dual space of \( H^{1/2}(\Gamma) \).

**Remark 1.6.** Note that

\[
H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)
\]

**Theorem 1.2.** (Trace Theorem in \( E(\Omega) \)): Let \( \Omega \) be an open bounded set of class \( C^2 \). Then there exists a continuous linear operator \( \gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma) \) such that \( \gamma_\nu u = u \cdot n \) on \( \Gamma \) for \( u \in D(\overline{\Omega}) \). Further, for any \( u \in E(\Omega) \) and scalar function \( w \in H^1(\Omega) \)

\[
\langle u, \nabla w \rangle + \langle \nabla \cdot u, w \rangle = \langle \gamma_\nu u, \gamma_0 w \rangle
\]

**Proof.** Let \( \phi \in H^{1/2}(\Gamma) \) and \( w \in H^1(\Omega) \) such that \( \gamma_0 w = \phi \). For \( u \in E(\Omega) \), define corresponding functional

\[
X_u(\phi) = (\nabla \cdot u, w) + (u, \nabla w) = \int_\Omega (w \nabla \cdot u + u \cdot \nabla w) \, dx
\]

First, we note that \( X_u(\phi) \) is independent of the particular choice of \( w \) since if we had \( \gamma_0 w_1 = \gamma_0 w_2 = \phi \), then the difference of two calculations of \( X_u(\phi) \) involving \( w_1 \) and \( w_2 \) respectively would be

\[
(\nabla \cdot u, w) + (u, \nabla w), \text{ where } w = w_1 - w_2
\]

Since \( \gamma_0 w = \gamma_0 w_1 - \gamma_0 w_1 = \phi - \phi = 0 \), we know that \( w \in H^1_0(\Omega) \) and can be approximated by a sequence \( w_m \in D(\Omega) \) and integration by parts gives

\[
(\nabla \cdot u, w_m) + (u, \nabla w_m) = 0
\]

Therefore, \( X_u(\phi) \) does not depend on the particular choice of \( w \in H^1(\Omega) \). We define one such \( w = l_\Omega \phi \). Then, from Cauchy-Schwartz,

\[
|X_u(\phi)| \leq \|u\|_{E(\Omega)} \|w\|_{H^1(\Omega)} \leq c_0 \|u\|_{E(\Omega)} \|\phi\|_{H^{1/2}(\Gamma)},
\]

where \( c_0 \) is the norm of the bounded linear operator \( l_\Omega \). Therefore the mapping \( \phi \to X_u(\phi) \) is a linear continuous mapping from \( H^{1/2}(\Gamma) \) to \( \mathbb{R} \). Thus, there exists \( g = g(u) \in H^{-1/2}(\Gamma) \) such that \( X_u(\phi) = \langle g, \phi \rangle \). Further, it is clear that

\[
\|g\|_{H^{-1/2}(\Gamma)} \leq c_0 \|u\|_{E(\Omega)}
\]

We define \( \gamma_\nu u = g(u) \), which is a continuous linear mapping from \( E(\Omega) \) into \( H^{-1/2}(\Gamma) \). Furthermore, for \( u, w \in D(\overline{\Omega}) \), then, using divergence theorem,

\[
X_u(\phi) = \int_\Omega \nabla \cdot (wu) \, dx = \int_\Gamma \phi u \cdot ndx = \langle u \cdot n, \gamma_0 w \rangle
\]
Since for these functions $w, \gamma_0 w$ form a dense set in $H^{1/2}(\Gamma)$, it follows that the formula $X_u(\phi) = \langle u \cdot n, \phi \rangle$ is true for any $\phi \in H^{1/2}(\Gamma)$ and therefore, $\gamma_\nu u = u \cdot n$ for $u \in D(\Omega)$.

**Remark 1.7.** The operator $\gamma_\nu$ actually maps $E(\Omega)$ onto $H^{-1/2}(\Gamma)$. To show this let $\phi \in H^{-1/2}(\Gamma)$ such that $\langle \phi, 1 \rangle = 0$. Then the Neumann problem
\begin{equation}
\Delta p = 0, \text{in } \Omega, \quad \frac{\partial p}{\partial n} = \phi, \text{ on } \Gamma
\end{equation}
has a weak solution $p = p(\phi) \in H^1(\Omega)$, which is unique up to an additive constant. For one of these solutions define $u = \nabla p$. It is clear that $u \in E(\Omega)$ and $\gamma_\nu u = \frac{\partial p}{\partial n} = \phi$. Further, we can easily construct $u_0 \in C^1(\Omega)$ with $\gamma_\nu u_0 = 1$. Then for any $\psi \in H^{1/2}(\Gamma)$, decomposing
\begin{equation}
\psi = \phi + \frac{\langle \psi, 1 \rangle}{\text{measure of } \Gamma},
\end{equation}
we can define $u$ depending on $\psi$ with $\gamma_\nu u = \psi$ simply by setting
\begin{equation}
u = \nabla p + \frac{\langle \psi, 1 \rangle}{\text{measure of } \Gamma} u_0,
\end{equation}
where $p$ is determined from (1.16) in terms of $\phi$. Moreover the mapping $\psi \to u(\psi)$ is a linear continuous mapping from $H^{-1/2}(\Gamma)$ into $E(\Omega)$, i.e. is a lifting operator $l_\Omega$.

**Definition 1.8.** Define $E_0(\Omega)$ to be the closure of $D(\Omega)$ in $E(\Omega)$.

We have the following theorem

**Theorem 1.3.** The kernel of $\gamma_\nu$ is equal to $E_0(\Omega)$.

**Proof.** If $u \in E_0(\Omega)$, then by definition, there exists a sequence $u_m \in D(\Omega)$ which converges to $u$ in $E(\Omega)$. Since $u_m = 0$ on boundary $\Gamma$, it follows that $\gamma_\nu u_m = 0$ and therefore $\gamma_\nu u = 0$.

Now assume for $u \in E(\Omega)$, $\gamma_\nu u = 0$. Since $u \in E(\Omega)$, it can be approximated arbitrarily closely by functions in $D(\Omega)$. Let $\Phi \in D(\mathbb{R}^n)$ and $\phi$ be the restriction of $\Phi$ to $\Omega$. Since $\gamma_\nu u = 0$, it follows that $\langle \gamma_\nu u, \gamma_0 \phi \rangle = 0$, implying
\begin{equation}
\int_\Omega (\phi \nabla \cdot u + u \cdot \nabla \phi) \, dx = 0
\end{equation}
Define for functions $v$ in $\Omega$ extensions to $\mathbb{R}^n$ so that $\overline{v}(x) = v(x)$ for $x \in \Omega$ and $\overline{v}(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then, we may re-write relation
We have
\[
\int_{\mathbb{R}^n} (\Phi \nabla \cdot \overline{u} + \overline{u} \cdot \nabla \Phi) \, dx = 0 = \int_{\mathbb{R}^n} (\Phi \nabla \cdot \overline{u} + \overline{u} \cdot \nabla \Phi) \, dx,
\]
for any $\Phi \in D(\mathbb{R}^n)$. This implies $\nabla \cdot \bar{u} = \nabla \cdot \overline{u}$ and therefore $\overline{u} \in E(\mathbb{R}^n)$.

Remark 1.9. If the set $\Omega$ is unbounded or if its boundary is not smooth, some partial results remain true: for example, if $u \in E(\Omega)$, we can define $\gamma_\nu u$ on each bounded part $\Gamma_0$ of $\Gamma$ of class $C^2$, and $\gamma_\nu u \in H^{-1/2}(\Gamma_0)$. If $\Omega$ is smooth but unbounded or if the boundary is the union of a finite number of bounded $(n-1)$-dimensional manifolds of class $C^2$, then $\gamma_\nu u$ is defined in this way on all $\Gamma$. Nevertheless, the generalized Stokes formula (1.9) does not hold.

Remark 1.10. If $u \in H^1(\Omega)$, then we will assume the following results as known: (a) There exists continuous linear mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ such that $\gamma_0 u = u \big|_\Gamma$ for every $u \in D(\Omega)$. We denote $H^{1/2}(\Gamma) = \gamma_0(H^1(\Omega))$; (b) There exists a lifting operator $l_\Omega : H^{1/2}(\Gamma) \to H^1(\Omega)$ such that $\gamma_0 \circ l_\Omega = I$, the identity operator on $H^{1/2}(\Gamma)$; (c) $\Omega$ is a Lipschitz set. Then, all the preceding results can be extended to this case. Theorems 1.1-1.3 hold. The proof of Theorem 1.2 leads to a definition of $\gamma_\nu u \in H^{-1/2}(\Gamma)$, which is the dual space of $H^{1/2}(\Gamma)$. The generalized Stokes formula (1.9) is also valid.