Week 12 Notes, Math 8610, Tanveer

1. Navier-Stokes with Boundaries

We now turn to analysis of Navier-Stokes equations with solid boundaries on which no-slip condition boundary conditions will be imposed. The notes in this part follow Temam's book on Navier-Stokes equations.

We will assume that the domain Ω is an open set on one side of a boundary Γ that is locally Lipschitz, *i.e.* locally in a neighborhood of a point $x \in \Gamma$, we have a representation $x_j = \theta\left(x_1, x_2, \cdots x_{j-1}, x_{j+1}, \cdots x_n\right)$ for some j, where θ is a Lipschitz function. It is known that such a a set Ω is locally star-shaped, *i.e.* for each point $x_j \in \Gamma$, there exists an open neighborhood $\mathcal{O}_j$ such that there $\mathcal{O}_j' = \Omega \cap \mathcal{O}_j$ is star-shaped with respect to one of its points. It is clear that if Γ is finite, then it has a finite open cover $\left\{\mathcal{O}_j\right\}_{j\in J}$ for a finite set J. We may arrange the same to be true even when Γ is not finite.

Definition 1.1. *Let $\mathcal{D}\left(\Omega\right)$ and be the space of C^∞ functions with compact support contained Ω.*

Definition 1.2. *Let $\mathcal{V}$ be the space of functions*

$$\mathcal{V} = \left\{u \in \mathcal{D}\left(\Omega\right)\ , \nabla \cdot u = 0\right\} \tag{1.1}$$

We define the closure of $\mathcal{V}$ in $L^2\left(\Omega\right)$ and $H_0^1\left(\Omega\right)$ to be H and V respectively.

2. Steady Navier-Stokes equation with stationary boundaries

First, we assume Ω to be a Lipschitz, bounded and open set in $\mathbb{R}^n$. We assume $f \in L^2(\Omega)$. We look for solution $u : \mathbb{R}^n \to \mathbb{R}^n$, $p : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \text{ in } \Omega \tag{2.2}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{2.3}$$

$$u = 0 \text{ on } \Gamma \tag{2.4}$$

If u, p and f were smooth functions satisfying (2.2)-(2.4), then clearly $u \in V$, and for any $v \in \mathcal{V}$, inner product of (2.2) with v and integration leads to

$$\nu\left(u, v\right)_1 + b(u, u, v) = (f, v)_0\ , \tag{2.5}$$

where recall that $(,)_m$ is the inner-product in the H^m Hilbert space and b is a trilinear functional of its arguments defined by

$$b(u,v,w) = \int_\Omega u_i v_{j,i} w_j dx \tag{2.6}$$

It is useful to identify the given $f \in L^2$ with $f \in V'$ the dual space of V, such that the right side of (2.7) may be interpreted as $\langle f, v\rangle$. We then rewrite (2.7) as

$$\nu\,(u,v)_1 + b(u,u,v) = \langle f, v\rangle \tag{2.7}$$

Equation (2.7) provides the the weak formulation of steady Navier-Stokes equation–we require that the solution $u \in V$ satisfies (2.7) for any $v \in V$ (note V is the closure of $\mathcal{V}$ in H_0^1 norm), where $f \in V'$ is a given function. It may be shown that any such solution to (2.7) satisfies (2.2) in a distributional sense.

3. Weak solutions to steady Navier Stokes

Our focus right now will be to prove solutions to (2.7) exist[(1)].

Lemma 3.1. *The trilinear form b is defined and trilinear continuous on $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ for $2 \le n \le 4$ and for bounded Ω.*

Proof. If $u, v, w \in H_0^1(\Omega)$, and $2 < n \le 4$,

$$u \in L^{2n/(n-2)}(\Omega) \ \ ,\ Dv \in L^2(\Omega) \ \ , w \in L^n(\Omega), \tag{3.8}$$

where we used Sobolev embedding theorems, definition of H_0^1, the observation that $n \le \frac{2n}{n-2}$ for $2 \le n \le 4$ and the fact that $L^{2n/(n-2)}(\Omega) \subset L^n(\Omega)$ for finite bounded domain Ω. By the Holder inequality,

$$\Big| \int_\Omega u_i v_{j,i} w_j dx \Big| \le c' \|u_i\|_{L^{2n/(n-2)}(\Omega)} \|v_{j,i}\|_{L^2(\Omega)} \|w_j\|_{L^n(\Omega)} \tag{3.9}$$

It follows that

$$\Big| b(u,v,w) \Big| \le c(n,\Omega) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)} \tag{3.10}$$

When $n = 2$, the same results (3.10) hold since

$$\Big| \int_\Omega u_i v_{j,i} w_j dx \Big| \le \|u_i\|_{L^4(\Omega)} \|v_{j,k}\|_{L^2(\Omega)} \|w\|_{L^4(\Omega)}, \tag{3.11}$$

and for $n = 2$ and Sobolev inequality gives $\|g\|_{L^4(\Omega)} \le c' \|g\|_{H^1(\Omega)}$. Continuity of the trilinear form b in each argument follows immediately from (3.10). ∎

[(1)] It is to be noted that if Ω is not bounded then b as defined in (2.6) need not make sense. In that case, we have to introduce auxiliary space for v other than V

Corollary 3.2. *For an open bounded set* Ω *and* $u, v, w \in V$ *and* $2 \leq n \leq 4$, b *is trilinear continuous form on* $V \times V \times V$.

Definition 3.3. *For* $u, v \in H_0^1(\Omega)$, *we define* $B(u, v)$ *the linear continuous form on* V *defined by*

$$\langle B(u,v), w\rangle = b(u,v,w) \ \ , u, v \in H_0^1(\Omega) \text{ for any } \ w \in V \tag{3.12}$$

Lemma 3.4. *For any open bounded set* Ω, *for* $2 \leq n \leq 4$,

$$b(u,v,v) = 0 \ \ , \text{for any } u \in V, v \in H_0^1(\Omega) \tag{3.13}$$

$$b(u,w,v) = -b(u,v,w) \ \ , \text{for any } u \in V, v, w \in H_0^1(\Omega) \tag{3.14}$$

Proof. It is enough to prove these equalities for $u \in \mathcal{V}$ and $v \in \mathcal{D}(\Omega)$ since they are dense in the given spaces. Integration by parts gives

$$b(u,v,v) = \int_\Omega u_j v_{i,j} v_i dx = \int_\Omega \partial_{x_j}\left(\frac{1}{2} u_j v_i v_i\right) dx = 0 \tag{3.15}$$

Now, if we replace v in (3.13) by $v + w$, we obtain

$$0 = b(u,v{+}w,v{+}w) = b(u,v,w){+}b(u,v,v){+}b(u,w,v){+}b(u,w,w) = b(u,v,w){+}b(u,w,v) \tag{3.16}$$

and (3.14) follows. ∎

Before we prove a theorem on existence of steady solutions, we will need the following preliminary lemma:

Lemma 3.5. *Let* X *be a finite dimensional Hilbert sapce with inner-product* $(\ ,\ )$ *and norm* $\Big|.\Big|$. *Let* P *be a continuous mapping from* X *to itself such that*

$$(P(\xi), \xi) > 0 \ \ , \text{for } \Big|\xi\Big| = k > 0 \tag{3.17}$$

Then, there exists $\xi \in X$ *with* $|\xi| \leq k$ *such that* $P(\xi) = 0$.

Proof. Suppose that P has no zero in the closed Ball $B \subset X$ centered at 0 with radius k. Then,

$$S(\xi) = -\frac{kP(\xi)}{\Big|P(\xi)\Big|} \tag{3.18}$$

maps the B back to itself and is continuous in ξ. Brower fixed point theorem implies that S has a fixed point in B, *i.e.* there exists $\xi_0 \in B$ so that

$$\xi_0 = -\frac{kP(\xi_0)}{\Big|P(\xi_0)\Big|} \tag{3.19}$$

Clearly from above $\left|\xi_0\right| = k$. Inner-product with ξ_0 and use of (3.17) leads to

$$(\xi_0, \xi_0) = -\frac{k\left(P(\xi_0), \xi_0\right)}{\left|P(\xi_0)\right|} < 0, \tag{3.20}$$

which is a contradiction. Hence we must have a zero of $P(\xi)$ for $|\xi| \leq k$. ∎

Lemma 3.6. *If $u_{(m)} \to u$ in V weakly and in $L^2(\Omega)$ strongly, then*

$$b(u_{(m)}, u_{(m)}, v) \to b(u, u, v) \ \text{, for any } v \in \mathcal{V} \tag{3.21}$$

Proof. We know that

$$b(u_{(m)}, u_{(m)}, v) = -b\left(u_{(m)}, v, u_{(m)}\right) = -\int_\Omega u_{(m),i} u_{(m),j} \partial_{x_i} v_j dx \tag{3.22}$$

Therefore, we have

$$b(u_{(m)}, u_{(m)}, v) - b(u, u, v) = -\int_\Omega u_{(m),i}\left(u_{(m),j} - u_j\right) v_{j,i} dx - \int_\Omega \left(u_{(m),i} - u_i\right) u_j v_{j,i} dx \tag{3.23}$$

Since $\|Dv\|_{L^\infty(\Omega)} < \infty$, and $\left\{u_{(m)}\right\}_m$ is a bounded sequence in $L^2(\Omega)$, application of Cauchy-Schwartz inequality completes the proof. ∎

Theorem 3.1. *(Existence of steady solution): Let Ω be a bounded set in $\mathbb{R}^n$ for $2 \leq n \leq 4$ and $f \in H^{-1}(\Omega)$. Then there exists at least one weak solution $u \in V$ to steady Navier Stokes equation (2.7).*

Proof. We will take as given the fact that Stokes operator $-\mathcal{P}\Delta$, where $\mathcal{P}$ is the Hodge projection, has an orthornormal set of eigen functions $\left\{w^{(i)}\right\}_{i=1}^\infty$ that form a complete set in V and that each $w^{(i)}$ is smooth in Ω. Then, for each fixed integer $m \geq 1$, we use a Galerkin approximation

$$u_{(m)} = \sum_{i=1}^m \xi_{i,m} w^{(i)} \tag{3.24}$$

We require choice of coefficients $\xi_{i,m}$, if there exists one, so that we satisfy the Galerkin approximation to (2.7):

$$\nu\left(u_{(m)}, w^{(k)}\right) + b(u_{(m)}, u_{(m)}, w^{(k)}) = \langle f, w^{(k)}\rangle \ \text{, for } k = 1, 2, \cdots, m \tag{3.25}$$

Equation (3.25) constitute a system of nonlinear equations for $\xi_{1,m}, \xi_{2,m}, ...\xi_{m,m}$. We will now prove that this nonlinear system has a solution. Define X to be the space spanned by $w^{(1)}$, $w^{(2)}$, .. $w^{(m)}$ and the inner product

in X will be the inner product $(,)_1$ induced by V, while $P = P_m$ is defined by
(3.26)
$(P_m(u), v) = (P_m(u), v)_1 = \nu(u, v)_1 + b(u, u, v) - \langle f, v\rangle$, for any $u, v \in X$

From properties of b in Lemma 3.4, it is clear that P_m is a continuous mapping in X, and we have

$$(P_m(u), u) = \nu\|u\|_1^2 - \|f\|_{V'}\|u\|_1 = \|u\|_1 \left(\nu\|u\|_1 - \|f\|_{V'}\right) \tag{3.27}$$

It follows that $(P_m(u), u) > 0$ for $\|u\| = k > \frac{1}{\nu}\|f\|_{V'}$. Using lemma 3.5, we know there exists a solution $u_{(m)}$ to (3.25). Now, we seek to determine limit of $m \to \infty$. If we multiply (3.25) by $\xi_{k,m}$ and sum over $k = 1, 2, \cdot, m$, it follows that

$$\nu\|u_{(m)}\|_1^2 + b\left(u_{(m)}, u_{(m)}, u_{(m)}\right) = \langle f, u_{(m)}\rangle \tag{3.28}$$

Since $b(u, u, u) = 0$, the above gives rise to the uniform estimate in m:

$$\|u_{(m)}\|_1 \leq \frac{1}{\nu}\|f\|_{V'} \tag{3.29}$$

Since the sequence $u_{(m)}$ remains bounded in V, by Banach-Alouglu theorem, there exists some $u \in V$ and a subsequence $m' \to \infty$ so that

$$u_{(m')} \to u \text{ , weakly in V,} \tag{3.30}$$

and therefore strongly in $L^2(\Omega)$ since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Now, we take $v = w_j$ for any fixed j, From (3.25) for $m \geq j$, it follows that

$$\nu\left(u_{(m)}, v\right) + b(u_{(m)}, u_{(m)}, v) = \langle f, v\rangle \tag{3.31}$$

Using Lemma 3.6, on the subsequence $u_{(m')}$, it follows that

$$\nu\,(u, v) + b(u, u, v) = \langle f, v\rangle \tag{3.32}$$

Since this is true for any $v = w_j$, it is also true for a linear combination of w_j and by density for $v \in V$, and we have always have a weak solution $u \in V$ to the Navier-Stokes equation. ∎

Theorem 3.2. *Uniqueness for large ν (small Reynolds number) Assume $2 \leq n \leq 4$ and domain Ω is Lipschitz and bounded. If ν is sufficiently large or equivalently $\|f\|_{V'}$ is sufficiently small, then there exists unique weak solution $u \in V$ to (2.7).*

Proof. In (2.7), we substitute $v = u$ to obtain

$$\nu\|u\|_1^2 = \langle f\, u\rangle \leq \|f\|_{V'}\|u\|_1 \text{ , implying } \|u\|_1 \leq \frac{1}{\nu}\|f\|_{V'} \tag{3.33}$$

Also, if $u_*, u_{**} \in V$ are two different solutions to (2.7), then it follows from subtraction that $w = u_* - u_{**}$ satisfies

$$\nu\,(w,v)_1 + b(w,u_*,v) + b(u_*,w,v) = 0 \tag{3.34}$$

Now choose $v = w$ and use Lemma 3.1 and (3.33) to obtain

$$\nu\|w\|_1^2 \le c(n,\Omega)\|w\|_1^2\|u_*\|_1 \le \frac{c(n,\Omega)}{\nu}\|f\|_{V'}\|w\|_1^2 \tag{3.35}$$

which gives rise to the only possibility $\|w\|_1 = 0$ when

$$\frac{c(n,\Omega)}{\nu^2}\|f\|_{V'} < 1 \tag{3.36}$$

∎