Week 13 Notes, Math 8610, Tanveer

1. Characterization of spaces H and V.

Recall spaces $\mathcal{V}$, H and V:

$$\mathcal{V} = \{v \in \mathcal{D}(\Omega), \nabla \cdot v = 0\} \tag{1.1}$$

and H is the closure of V in $L^2(\Omega)$, while V is the closure in $H_0^1(\Omega)$.

1.1. **Characterization of gradient of a distribution.** Let $\Omega \subset \mathbb{R}^n$ be open and let p be a distribution on Ω, *i.e.* using standard notation $p \in \mathcal{D}'(\Omega)$. We note that for any $v \in \mathcal{V}$,

$$\langle \nabla p, v\rangle = \langle \partial_{x_j} p, v_j\rangle = -\langle p, \partial_{x_j} v_j\rangle = 0 \tag{1.2}$$

The converse of this result is also true as in the following proposition, though the proof is harder and will be skipped:

Proposition 1.1. *Let Ω be an open set of $\mathbb{R}^n$ and $f = (f_1, f_2, .., f_n)$, with $f_i \in \mathcal{D}'(\Omega)$. A necessary and sufficient condition that $f = \nabla p$ for some $p \in \mathcal{D}'(\Omega)$ is that*

$$\langle f, v\rangle = 0 \text{ ,for any } \; v \in \mathcal{V} \tag{1.3}$$

Definition 1.2. *We define the following subspace of $L^2(\Omega)$:*

$$L^2(\Omega)/\mathbb{R} = \left\{ p \in \mathcal{L}^2(\Omega), \int_\Omega p(x)dx = 0 \right\} \tag{1.4}$$

We will also use the following results without proof (See Deny & Lions *Ann. Inst. Fourier* **5**, 1954, pp 305-370.)

Proposition 1.3. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$.*
i. *If a distribution p has all its first-order derivatives $\partial_{x_i} p \in L^2(\Omega)$, then $p \in L^2(\Omega)$, which may be chosen to be in the subspace (1.4) with*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \le c(\Omega) \|\nabla p\|_{L^2(\Omega)} \tag{1.5}$$

ii. *If a distribution p has all its first derivatives $\partial_{x_i} p \in H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \le c(\Omega) \|\nabla p\|_{H^{-1}(\Omega)} \tag{1.6}$$

In both cases, if Ω is any open set in $\mathbb{R}^n$, $p \in L^2_{loc}(\Omega)$.

Remark 1.4. Combining results in the previous two propositions we note that if $f \in H^{-1}(\Omega)$ (or $L^2_{loc}(\Omega)$) and $\langle f, v\rangle = 0$ for any $v \in \mathcal{V}$, then $f = \nabla p$ with $p \in L^2_{loc}(\Omega)$. Moreover, if Ω is open Lipschitz bounded, then $p \in L^2(\Omega)$ (or $H^1(\Omega)$)

2. Characterization of the space H

We can now give the following characterization of H and $H^{\perp}$.

Theorem 2.1. *Let Ω be Lipschitz open bounded set in $\mathbb{R}^n$. Then*

$$H^{\perp} = \left\{ u \in L^2(\Omega), u = \nabla p, p \in H^1(\Omega) \right\} \tag{2.7}$$

$$H = \left\{ u \in L^2(\Omega), \nabla \cdot u = 0, \gamma_\nu u = 0 \right\} \tag{2.8}$$

Proof. Assume u is in the space on the right side of (2.7). Then for any $v \in \mathcal{V}$,

$$(v, u) = (v, \nabla p) = -(\nabla \cdot v, p) = 0 \tag{2.9}$$

Since $\mathcal{V}$ is dense in H, $u \in H^{\perp}$. Conversely, assume $u \in H^{\perp}$. Then for any $v \in \mathcal{V}$,

$$(u, v) = 0 \tag{2.10}$$

implying from Propositon 1.1 that $u = \nabla p$ for $p \in \mathcal{D}'(\Omega)$. By proposition 1.3, $p \in H^1(\Omega)$ and therefore u belongs to the space on the right hand side of (2.7).

Assume now that $u \in H$. Then $u = \lim_{m\to\infty} u_m$ in $E(\Omega)$, where $u_m \in \mathcal{V}$. Further $\gamma_\nu u = \lim_{m\to\infty} \gamma_\nu u_m = 0$ since $\gamma_\nu u_m = u_m \cdot n = 0$ for any $u_m \in \mathcal{V}$, recalling $\mathcal{V} \subset \mathcal{D}(\Omega)$. Hence $u \in H_*$, the space on the right side of (2.8), implying $H \subset H_*$. Assume H_{**} is the orthogonal compliment of H in H_*. Then from (2.7), it follows that for $u \in H_{**}$, there exists $p \in H^1(\Omega)$ with $u = \nabla p$ and moreover from (2.8), it follows that

$$\Delta p = \nabla \cdot u = 0 \ , \ u \cdot n = \frac{\partial p}{\partial n} = 0 \tag{2.11}$$

implying p is a constant (from Remark 1.7, in extra Week 12 notes), and therefore $u = 0$. This implies $H_* = H$ ∎

Theorem 2.2. *Let Ω be an open bounded set of class C^2. Then,*

$$L^2(\Omega) = H \oplus H_1 \oplus H_2 \tag{2.12}$$

$$H_1 = \left\{ u \in L^2(\Omega) \ , u = \nabla p \ , p \in H^1(\Omega) \ , \Delta p = 0 \right\} \tag{2.13}$$

$$H_2 = \left\{ u \in L^2(\Omega) \ , u = \nabla p \ , p \in H_0^1(\Omega) \right\} \tag{2.14}$$

Proof. Clearly from the characterization of $H^{\perp}$ in 2.1, $H_1, H_2 \subset H^{\perp}$. Now, take $u \in H_1$ and $v \in H_2$, with $v = \nabla p$. Then from generalized Stokes formula

$$(u, v) = (u, \nabla p) = \langle \gamma_\nu u, \gamma_0 p \rangle - (\nabla \cdot u, p) = 0 \tag{2.15}$$

Therefore, $H_1 \perp H_2$. Now, we seek to show that any $u \in L^2$ can be written as a sum of u_0, u_1 and u_2 in H, H_1 and H_2 respectively. Define $u_2 = \nabla p$, where $p \in H_0^1$ to be the unique solution of

$$\Delta p = \nabla \cdot u \in H^{-1}(\Omega) \tag{2.16}$$

We then define $u_1 = \nabla q$, where $q \in H^1(\Omega)$ is the solution (unique upto additive constant, see Remark 1.7 week 12 extra notes) to the Neumann problem

$$\Delta q = 0 \ , \frac{\partial q}{\partial n} = \gamma_\nu \left(u - \nabla p\right), \tag{2.17}$$

which exists since $\nabla \cdot (u - \nabla p) = 0$ and therefore $u - \nabla p \in E(\Omega)$ and so $\gamma_\nu \left(u - \nabla p\right) \in H^{-1/2}(\Gamma)$ is well-defined (Note Theorem 1.2 of Week 12 extra notes). Now, consider $u_0 = u - u_1 - u_2$. We note that $\nabla \cdot u_0 = \nabla \cdot u - \nabla \cdot u_1 - \nabla \cdot u_2 = 0$ and $\gamma_\nu u_0 = \gamma_\nu \left(u - \nabla p\right) - \frac{\partial q}{\partial n} = 0$. Therefore $u_0 \in H$. ∎

Remark 2.1. We may define $\mathcal{P}$ to be the orthogonal projection (Hodge projection) of $L^2(\Omega)$ onto H; clearly $\mathcal{P}$ is continuous into $L^2(\Omega)$. In fact $\mathcal{P} \to H^1(\Omega) \to H^1(\Omega)$ and is continuous in the norm of H^1. In the proof of the previous theorem, let us assume $u \in H^1(\Omega)$; then $p \in H_0^1(\Omega) \cap H^2(\Omega)$; and $u - \nabla p \in H^1(\Omega)$ and $\gamma_\nu(u - \nabla p) \in H^{1/2}(\Omega)$. Finally, we infer from (2.17) that $q \in H^2(\Omega)$ and

$$\mathcal{P}u = u - \nabla(p + q) \in H^1(\Omega) \tag{2.18}$$

It is also clear that the mappings $u \to p$ and $u - \nabla p \to q$ are continuous in the appropriate spaces and we conclude that $\mathcal{P} : H_0^1(\Omega) \to H^1(\Omega$ is continuous with

$$\|\mathcal{P}u\|_{H^1(\Omega)} \leq c(\Omega) \|u\|_{H^1(\Omega)} \tag{2.19}$$

If Ω is C^{r+1} for integer $r \geq 1$, a similar argument shows that for $u \in H^r(\Omega)$, $\mathcal{P}u \in H^r(\Omega)$ and $\mathcal{P}$ is linear and continuous with

$$\|\mathcal{P}u\|_{H^r(\Omega)} \leq c(r, \Omega) \|u\|_{H^r(\Omega)} \tag{2.20}$$

3. Characterization of space V

Theorem 3.1. *Let Ω be an open bounded Lipschitz set. Then*

$$V = \left\{ u \in H_0^1(\Omega) \ , \nabla \cdot u = 0 \right\} \tag{3.21}$$

Proof. Define V_* to be the set on the right side of (3.21). It is clear that $V \subset V_*$ since V is the closure of $\mathcal{V}$ in H_0^1. To prove $V = V_*$, it is enough to show that a continuous linear functional L in V_*, which vanishes on V, is identically 0. We know there exists $l \in V_*'$ so that for any $v \in V_*$,

$$L(v) = \langle l, v \rangle \tag{3.22}$$

Since V_* is a closed subspace of $H_0^1(\Omega)$ it may be extended as a linear continuous functional on $H_0^1(\Omega)$ and so $l \in H^{-1}(\Omega)$ and we have $\langle l, v \rangle = 0$ for any $v \in \mathcal{V}$. Propositions 1.1 and 1.3 imply that $l = \nabla p$ and $p \in L^2(\Omega)$, and thus for any $v \in H_0^1(\Omega)$,

$$L(v) = \langle l, v \rangle = \langle \partial_{x_i} p, v_i \rangle = -\left(p, \partial_{x_i} v_i\right) \tag{3.23}$$

Therefore $L(v) = 0$ for $v \in V_*$. ▮

4. Existence and uniqueness for the steady Stokes equations

let Ω be an open bounded set in $\mathbb{R}^n$ with boundary Γ and let $f \in L^2(\Omega)$. We seek to find solution to

$$-\nu \Delta u + \nabla p = f \text{ in } \Omega \tag{4.24}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{4.25}$$

$$u = 0 \text{ on } \Gamma \tag{4.26}$$

If f, u and p were smooth functions, it would follow from multiplying (4.24) and using integration by parts that for any $v \in \mathcal{V}$,

$$\nu \left(u, v\right)_1 = \left(f, v\right) \ , \tag{4.27}$$

where $(u, v)_1 = (Du, Dv)$. Since each side of (4.27) depends linearly and continuously on $v \in H_0^1(\Omega)$, the equality (4.27) holds for any $v \in V$, the closure of $\mathcal{V}$ in $H_0^1(\Omega)$. If Ω is of class C^2, then any smooth solution $u \in H_0^1(\Omega)$. From (4.25), $u \in V$ in addition to satisfying (4.27). We will now show that (4.27) provides for a weak formulation of Stokes equation (4.24)-(4.26).

Lemma 4.1. *Let Ω be an open bounded set of class C^2. The following conditions are equivalent*

i. *$u \in V$ and satisfies (4.27) for every $v \in V$.*

ii. *$u \in H_0^1(\Omega)$ satisfies (4.24)-(4.26) in the following weak sense: there exists $p \in L^2(\Omega)$ so that (4.24)-(4.25) are satisfied in the sense of distribution and $\gamma_0 u = 0$*

Proof. If (**ii.**) is satisfied, then (4.27) follows for any $v \in \mathcal{V}$ since we can make (4.24) act on v in the sense of a distribution. Since $\mathcal{V}$ is dense in V in H_0^1, statement (**i**) follows.

Assume now $u \in V$ satisfies (4.27) for any $v \in \mathcal{V}$. Then, it follows that

$$\langle -\nu\Delta u - f, v\rangle = 0 \text{ for any } \ v \in \mathcal{V} \tag{4.28}$$

From Proposition 1.1 and 1.3, there exists $p \in L^2(\Omega)$ such that

$$-\nu\Delta u - f = -\nabla p \tag{4.29}$$

in the sense of distribution. Further since $u \in V \subset H_0^1$, $\nabla \cdot u = 0$ in the sense of distribution and $\gamma_0 u = 0$; thus statement (**ii**) follows. ∎

Theorem 4.1. *(Projection Theorem) For any open set $\Omega \subset \mathbb{R}^n$, which is bounded in some direction and every $f \in L^2(\Omega)$, the problem (4.27) has unique solution $u \in V$ (the result is equally valid for any $f \in H^{-1}(\Omega)$). More over, there exists a function $p \in L^2_{loc}(\Omega)$ such that (4.24)-(4.25) are satisfied in the sense of distribution.*

If Ω is a bounded open set of class C^2, then $p \in L^2(\Omega)$ and (4.24)-(4.25) are satisfied by u and p.

This theorem is a consequence of Lemma 4.1 and the following (Lax-Milgram) Theorem:

Theorem 4.2. *If W is a separable real Hilbert Space with norm $\|.\|_W$ and let $a(u, v)$ be a linear continuous form on $W \times W$, so that there exists $\alpha > 0$ such that*

$$a(u, u) \geq \alpha \|u\|_w^2 \ \text{, for any } u \in W \tag{4.30}$$

Then for each $l \in W'$, the dual space of W, there exists unique $u \in W$ such that

$$a(u, v) = \langle l, v\rangle \quad \text{, for any } v \in W \tag{4.31}$$

To apply the above Theorem to prove Theorem 4.1, we take $W = V$ and $a(u, v) = (u, v)_1$ and define l so that $\langle l, v\rangle = (f, v)$. The space V is separable as a closed subspace of a separable space $H_0^1(\Omega)$ (in particular $V = \mathcal{P}H_0^1$).

Proof of Theorem 4.2 Uniqueness part of the theorem follows from the observation that if u_1 and u_2 are two solutions then

$$a(u_1 - u_2, v) = a(u_1, v) - a(u_2, v) = 0 \tag{4.32}$$

Using $v = u_1 - u_2$, we obtain $0 \geq \alpha\|u_1 - u_2\|_W^2$, implying $u_1 = u_2$. Now consider existence question. Since W is separable, thee exists $\{w_i\}_{i=1}^\infty$ which forms a basis for W. Define $W_m = Span\,\{w_1, w_2, \cdots, w_m\} \subset W$. For each fixed integer $m \geq 1$, we define approximate solution u_m:

$$u_m = \sum_{i=1}^m \xi_{i,m} w_i \tag{4.33}$$

and require it to satisfy

$$a(u_m, v) = \langle l, v\rangle \quad \text{, for any } v \in W_m \tag{4.34}$$

By replacing v by w_j, $j = 1, ..m$, we get a linear system of m equations for unknowns $\{\xi_{i,m}\}_{i=1}^{m}$:

$$a(u_m, w_j) = \langle l, w_j\rangle \text{ , for } j = 1, 2\cdots m \tag{4.35}$$

We now claim there exists a unique solution to (4.35) for if there were two such solutions their difference v_m would satisfy

$$a\,(v_m, w_j) = 0 \tag{4.36}$$

It follows that $a(v_m, v_m) = 0$, which implies $v_m = 0$. Therefore, we have a uniqe solution to (4.34). Now, by substituting $v = u_m$ in (4.34), it follows that

$$\alpha\|u_m\|_W^2 \le \|l\|_{W'}\|u_m\|_W \text{ , implying } \|u_m\|_W \le \frac{1}{\alpha}\|l\|_{W'} \tag{4.37}$$

Therefore, there exists subsequence $u_{m'} \to u \in W$ weakly. Take $v \in W_j$ to be a fixed element for some j. Then when $m' \ge j$, $v \in W_{m'}$ and according to (4.34)

$$a(u_m', v) = \langle l, v\rangle \tag{4.38}$$

On using the following Lemma 4.2 and taking the limit $m' \to \infty$, we obtain

$$a(u, v) = \langle l, v\rangle \tag{4.39}$$

Since this is true for $v \in W_j$ for any j we have

$$a(u, v) = \langle l, v\rangle \text{ for any } v \in W \tag{4.40}$$

Lemma 4.2. *Let $a(u, v)$ be a bilinear continuous form on a Hilbert space W. Let ϕ_m (or ψ_m) be a sequence of elements of W which converges to ϕ (or ψ) in the weak (or strong) topology of W. Then*

$$\lim_{m\to\infty} a\,(\psi_m, \phi_m) = a(\psi, \phi) \tag{4.41}$$

$$\lim_{m\to\infty} a\,(\phi_m, \psi_m) = a(\phi, \phi) \tag{4.42}$$

Proof. We write

$$a(\psi_m, \phi_m) - a(\psi, \phi) = a(\psi_m - \psi, \phi_m) + a(\psi, \phi_m - \phi) \tag{4.43}$$

Since a is continuous and the sequence ϕ_m is bounded

$$\Big|a\,(\psi_m - \psi, \phi_m)\Big| \le c\|\psi_m - \psi\|_W\|\|\phi_m\|_W \le c'\|\psi_m - \psi\|_W \tag{4.44}$$

and this term converges to 0 as $m \to \infty$.

We notice next that $v \to a(\psi, v)$ is continuous on W and there exists an element $A(\psi) \in W'$ with $a(\psi, v) = \langle A(\psi), v\rangle$ for every $v \in W$. Therefore, we can write

$$\lim_{m\to\infty} a(\psi, \phi_m - \phi) = \lim_{m\to\infty} \langle A(\psi), \phi_m - \phi\rangle = 0 \tag{4.45}$$

To prove the second statement (4.42), we simply use $a^*(u, v) = a(v, u)$. ∎

5. Eigenfunctions of the Stokes problem

Let Ω be an open bounded set in $\mathbb{R}^n$. Define $\Lambda : f \to \frac{1}{\nu} u$ to be the map defined in Theorem 4.1, which is clearly a linear and continuous map from $L^2(\Omega)$ onto $V \subset H_0^1(\Omega)$. Since Ω is bounded, the injection $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, implying the compactness of the operator Λ. Further, the Λ is self-adjoint and positive operator since

$$(\Lambda f_1, f_2) = \nu\,(u_1, u_2)_1 = (f_1, \Lambda f_2) \tag{5.46}$$

It follows that there exists orthonormal sequence of eigenfunctions $\{w_j\}_{j=1}^{\infty} \in V$ and corresponding set $\{\lambda_j\}_{j=1}^{\infty}$ of eigenvalues with $\lambda_j > 0$ and $\lambda_j \to \infty$ such that $\Lambda w_j = \frac{1}{\lambda_j} w_j$ and for any $j \geq 1$,

$$(w_j, v)_1 = \lambda_j\,(w_j, v) \text{ for any } v \in V \tag{5.47}$$

$$(w_j, w_k) = \delta_{j,k} \tag{5.48}$$

$$(w_j, w_k)_1 = \lambda_j \delta_{j,k} \tag{5.49}$$

Using Theorem 4.1, it follows that for any $j \geq 1$, there exists $p_j \in L^2(\Omega)$ so that

$$-\,\nu \Delta w_j + \nabla p_j = \lambda_j w_j \text{ in } \Omega \tag{5.50}$$

$$\nabla \cdot w_j = 0 \text{ in } \Omega \tag{5.51}$$

$$\gamma_0 w_j = 0 \tag{5.52}$$

These are the eigenfunctions of the Stokes problem. If Ω is of class C^m for integer $m \geq 2$, a iterated application of Proposition 1.3 shows that

$$w_j \in H^m(\Omega)\ \ , p_j \in H^{m-1}(\Omega)\ \ \text{, for any }\ \ j \geq 1 \tag{5.53}$$

If Ω is of class C^∞, then $w_j, p_j \in C^\infty(\overline{\Omega})$. and the asymptotic behavior $\lambda_j \sim c j^{n/2}$ can be derived from a variational approach.