1. **Introduction**

First, we describe the basic equations governing motion of fluids. The description is on a length scale far larger than inter-molecular distance and on a time scale far larger than the typical time between collision of molecules. Over such scales, it is appropriate to assume a *continuum* description of fluid mechanics, as described below. By averaging over many fluid molecules and many collisions, one avoids dealing with randomness of molecular motion.

In the *continuum* hypothesis, we postulate existence of appropriately smooth density (ρ), velocity ($\mathbf{u}$) and pressure p as a function of space and time. These are denoted by $\rho(\mathbf{x}, t)$, $\mathbf{v}(x, t)$, and $p(\mathbf{x}, t)$. In MKS system, these are measured in units of Kg/m^3, m/s and $Newton/m^2$ (or $Kg/m/sec$), respectively. Additionally, for compressible fluids, it also necessary to include additional dependent variables: temperature $T(\mathbf{x}, t)$, entropy $s(\mathbf{x}, t)$, enthalpy $w(\mathbf{x}, t)$ and internal energy $e(\mathbf{x}, t)$. The existence of such well defined smooth functions reduces fundamental laws of conservation of mass, momentum and energy into partial differential equations. In this course, we will derive these PDEs, discuss their solutions and sometimes discuss physical insights into actual behavior of fluids.

We will assume that the fluid occupies some open connected set $\Omega \subset \mathbb{R}^3$. Sometimes we will consider $\Omega = \mathbb{R}^3$, while time $t \in \mathbb{R}^+$. In special cases, when there is no dependence on one component of $\mathbf{x}$, say x_3, it is appropriate to define a reduced problem, where $\mathbf{x} = (x_1, x_2)$. In that case, the space domain $\Omega \subset \mathbb{R}^2$.

2. **Conservation of mass: continuity equation**

The principle of conservation of mass states *mass is neither created nor destroyed*. Consider fluid occupying a fixed closed and bounded region $W \subset \Omega^{(1)}$ The mass of a small infinitesimal volume $d\mathbf{x}$ at a point $\mathbf{x} \in W$ at time t is $\rho(\mathbf{x}, t)d\mathbf{x}$. The total mass m of fluid within W is therefore

$$m = \int_W \rho(\mathbf{x}, t) d\mathbf{x} \tag{2.1}$$

Note, in general m depends on time t.

If we assume that there is no *source* or *sink* of fluid in the region W, then m changes with time *only* because fluid moves in or out of the region W.

Consider a infinitesimal surface element of area dA at $\mathbf{x}$ on the surface of W (see Fig. 1), with outward normal $\mathbf{n}$.

The volume of fluid getting out of W through dA in unit time must be $\mathbf{u} \cdot \mathbf{n} dA$ (see Fig. 2). Therefore, the mass of fluid getting out of W through dA, called the outward *mass-flux*, is $\rho \mathbf{u} \cdot \mathbf{n} dA$. The total *mass-flux* out of W must be

$$\int_{\partial W} \rho \mathbf{u} \cdot \mathbf{n} dA$$

From principle of *conservation of mass*, the rate of decrease of mass m must be equal to the outward flux. Therefore,

$$-\frac{dm}{dt} = \int_{\partial W} \rho \mathbf{u} \cdot \mathbf{n} dA \tag{2.2}$$

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(1) For now, we will assume W to be a simply connected set in $\mathbb{R}^3$, with a smooth boundary ∂W, though smoothness assumption can be weakened.

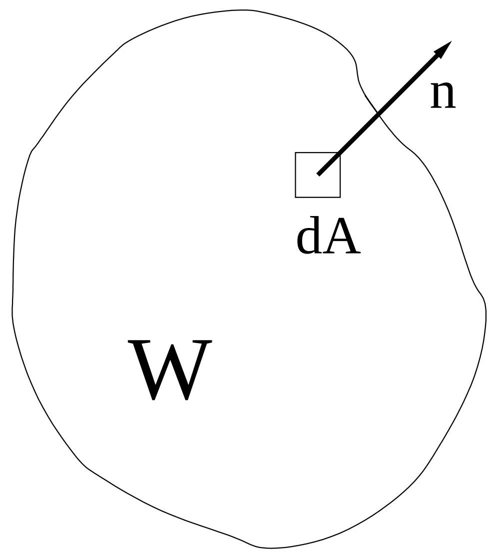

FIGURE 1. $W \subset \Omega \subset \mathbb{R}^3$, with normal $\mathbf{n}$ at $\mathbf{x} \in \partial\Omega$

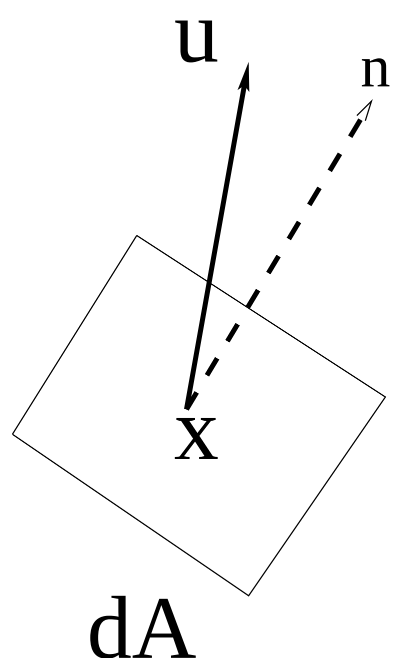

FIGURE 2. Volume escaping W through dA at $\mathbf{x} \in \partial\Omega$ is $\mathbf{u} \cdot \mathbf{n} dA$

Using (2.1) and (2.2), appropriate regularity of ρ, $\mathbf{u}$ and ∂W and Gauss's divergence Theorem,

$$\int_W \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right\} d\mathbf{x} = 0 \tag{2.3}$$

Since (2.3) is true for arbitrary W, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.4}$$

Using vector identities, we may rewrite (2.3) as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \tag{2.5}$$

Equation (2.3) or (2.5) is referred to as the general *continuity* equation and follows from conservation of mass. Note, this is valid without any assumption on whether fluid is compressible or not.

3. **Balance of momentum**

Consider the momentum within a fixed volume W. Since momentum is the product of mass and velocity, for an infinitesimal volume $d\mathbf{x}$, the corresponding momentum is $\mathbf{u}\rho d\mathbf{x}$. Therefore, the total momentum within W is

$$\mathbf{M} = \int_W \rho \mathbf{u} d\mathbf{x} \tag{3.6}$$

Now momentum within the fixed volume W changes because (i) fluid with momentum moves in and out of the volume W and (ii) force causes a change in momentum according to Newton's second law.

Consider (i) the momentum transferred in and out of W to neighboring fluid. Realizing the vector nature of momentum, it is convenient to consider the scalar: i-th component of the momentum, where $i = 1, 2$ or 3. Similar to expression for outward flux of mass (see Fig. 2), the outward flux of the i-th component of momentum through area element dA is given by $(\rho u_i)\mathbf{u} \cdot \mathbf{n} dA$. So, the net outward flux $\mathcal{F}_{M,i}$ of i-th component of momentum is given by

$$\mathcal{F}_{M,i} = \int_{\partial W} (\rho u_i) \mathbf{u} \cdot \mathbf{n} dA = \int_W \nabla \cdot [\rho u_i \mathbf{u}]\, dx = \int_W \{\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot [\rho \mathbf{u}]\} \tag{3.7}$$

As a vector with components $\mathcal{F}_{M,i}$, we have

$$\mathcal{F}_M = \int_W \{\rho [\mathbf{u} \cdot \nabla] \mathbf{u} + \mathbf{u} \nabla \cdot [\rho \mathbf{u}]\} \tag{3.8}$$

Now, consider source (ii) for change of momentum, which is the *external* force acting on W. Note different parts of the fluid within volume W exert force on each other; however, from Newton's third law, these *internal* forces are equal and opposite to each other and therefore do not contribute to the net force on W. There are two kind of external forces on W. The first type are (a) body forces, like gravity or electro-magnetic forces, if they are relevant. These act at every point W. We will denote it by $\mathbf{b}(\mathbf{x}, t)$ per unit mass that the total body force on W is

$$\mathbf{F}_b = \int_W \mathbf{b} \rho d\mathbf{x} \tag{3.9}$$

The second kind of force on W is due to (b) surface forces, which act only on boundary ∂W. For an *ideal* fluid, defined by neglect of *viscous-friction*[(2)], surface force on W acts towards the inward normal $-\mathbf{n}$. The magnitude of this surface

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(2) This friction is an aggregate effect resulting from molecular collisions that tend to slow down fast moving fluid-molecules and speed up slower ones

force per unit area is called pressure p. Therefore, from definition of pressure, the surface force on W for an *ideal* fluid is:

$$\mathbf{F}_S = \int_{\partial W} [-p\mathbf{n}] dA = -\int_W \nabla p d\mathbf{x}, \text{ from divergence theorem} \tag{3.10}$$

From balance of momentum, implicit in Newton's second law,

$$\frac{d}{dt}\mathbf{M} = -\mathcal{F}_M + \mathbf{F}_b + \mathbf{F}_S \tag{3.11}$$

Note that the minus sign on the first term on the right is because $\mathcal{F}_M$ measures the *outward* momentum flux; if $\mathcal{F}_{M,i} > 0$, $\frac{d}{dt}M_i < 0$ when $\mathbf{F}_b + \mathbf{F}_S = 0$. Using expressions for $\mathbf{M}$, $\mathcal{F}_M$, $\mathbf{F}_b$ and $\mathbf{F}_S$ in (3.6), (3.8), (3.9) and (3.10), the latter only valid for an *ideal* fluid,

(3.12)
$$\int_W \frac{\partial}{\partial t}[\rho\mathbf{u}] \; d\mathbf{x} \; = \; -\int_W \{\rho[\mathbf{u}\cdot\nabla]\mathbf{u} + \mathbf{u}\nabla\cdot[\rho\mathbf{u}]\} \;\; d\mathbf{x} + \int_W \rho\mathbf{b} d\mathbf{x} - \int_W \;\nabla p \; d\mathbf{x}$$

Since, this is true for any volume W, it follows that for an *ideal* fluid, momentum balance implies:

$$\frac{\partial}{\partial t}[\rho\mathbf{u}] + \rho[\mathbf{u}\cdot\nabla]\mathbf{u} + \mathbf{u}\nabla\cdot[\rho\mathbf{u}] = \rho\mathbf{b} - \nabla p \tag{3.13}$$

We note that

$$\frac{\partial}{\partial t}[\rho\mathbf{u}] + \rho[\mathbf{u}\cdot\nabla]\mathbf{u} + \mathbf{u}\nabla\cdot[\rho\mathbf{u}] = \mathbf{u}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\mathbf{u}]\right) + \rho\left(\frac{\partial\mathbf{u}}{\partial t} + [\mathbf{u}\cdot\nabla]\mathbf{u}\right)$$

Hence using *continuity equation* (2.5), momentum equation (3.13) for *ideal* fluid reduces to

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \rho\mathbf{b} \tag{3.14}$$

We note since $\mathbf{F}_b + \mathbf{F}_S$ is the total force on W and the volume of W is $\int_{x\in W} d\mathbf{x}$, it follows from (3.8) and (3.9), that the right hand side of (3.14) is the Force per unit volume acting at $\mathbf{x}$ at time t. This gives an alternate interpretation of (3.14). Consider a fluid particle[(3)] located at $\mathbf{x} = \mathbf{x}(t)$. The velocity of this particle is $\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t), t)$ from definition of $\mathbf{u}$. However, we note that its acceleration

$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{d}{dt}\mathbf{u}(\mathbf{x}(t), t) = \frac{\partial}{\partial t}\mathbf{u}(\mathbf{x}, t) + \frac{d\mathbf{x}}{dt}\cdot\nabla\mathbf{u}(\mathbf{x}, t) = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \equiv \frac{D\mathbf{u}}{Dt} \tag{3.15}$$

So, (3.14) may be written as

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b} \tag{3.16}$$

In this form (3.16) is see as a direct statement of Newton's second law: the product of *mass* times acceleration per unit volume equals total force per unit volume.

[(3)] Not to be confused with a fluid molecule. We using *continuum* hypothesis; so any graininess is being neglected.

3.1. **Viscous Effects.** As mentioned earlier, the expression $\mathbf{F}_S$ in (3.10) and hence the momentum equations, either in the form (3.14) or (3.16), are valid only when we assume an *ideal fluid*, *i.e.* a fluid where surface force acts in the inward normal direction. This ignores the fact that layers of fluids moving parallel to each other will cause friction. More realistically, one includes a *Viscous Stress* tensor $\mathbf{T}$, defined such that the i-th component of the total surface force $\mathbf{F}_S$ on the area element dA, is given by

$$dF_{S,i} = \left\{ -pn_i + \sum_{j=1}^{3} T_{i,j} n_j \right\} dA \tag{3.17}$$

Here $\mathbf{T}$ is a second order tensor, with components $T_{i,j}$. Therefore, (3.10) is replaced by

$$\mathbf{F}_S = \int_W \{-\nabla p + \nabla \cdot \mathbf{T}\}\, d\mathbf{x} \tag{3.18}$$

where we define vector $\nabla \cdot \mathbf{T}$ so that $[\nabla \cdot \mathbf{T}]_i = \sum_{j=1}^{3} \partial_{x_j} T_{i,j}$. Therefore, the momentum equation (3.14) becomes

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{3.19}$$

For so-called *Newtonian* fluid, which includes most common fluids[(4)], $\mathbf{T}$ is determined in terms of *strain-tensor* $\mathbf{S}$[(5)], whose components are

$$S_{i,j} = \frac{1}{2} \left(\partial_{x_j} u_i + \partial_{x_i} u_j \right) \tag{3.20}$$

The relation between *stress* and *strain* is given by a *constitutive* relation, which models the physical behavior of fluids as seen in experiment. For *Newtonian* fluid, this is given by

$$\mathbf{T} = 2\mu \left(\mathbf{S} - \frac{1}{3}(\nabla \cdot \mathbf{u})\mathbf{I} \right) + \left[\lambda + \frac{2}{3}\mu \right] (\nabla \cdot \mathbf{u})\mathbf{I} \tag{3.21}$$

where μ and λ are constants, referred to as *first* and and *second* coefficient of *viscosity*[(6)]. When (3.21) is used in (3.19), vector identities show that the momentum equation becomes:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \rho \mathbf{b} \tag{3.22}$$

4. **Incompressible Fluid Equations**

Recall that actual fluid-particle motion (or *Langrangian* motion) of is governed by the ODE:

$$\frac{d\boldsymbol{\xi}}{dt} \;=\; \mathbf{u}(\xi(t), t)\ ,\ \xi(0) = x \tag{4.23}$$

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(4) This includes air, water; but not tooth-paste or blood which are *Non-Newtonian* fluid

(5) *Strain tensor* is the symmetric part of $\nabla \mathbf{u}$. The anti-symmetric part is related to simply rotation of fluid element; this cannot contribute to fluid stresses

(6) Most often in the literature viscosity coefficient refers to μ since λ is not relevant for incompressible flows, as we shall seee later

We call this solution $\phi(\mathbf{x}, t)$ This can be used as a map of an initial region W. We define

$$W_t \equiv \{\xi : \xi = \phi(\mathbf{x}, t)\ ,\ \xi(0) = x \in W\}\,, \tag{4.24}$$

which physically corresponds to the *same* set of particles that initially constitutes W, even as they move around with time. For smooth velocity fields $\mathbf{u}$, W_t is clearly a smooth map of W. At each time t, at point $x \in W$, we can define the Jacobian J of this transformation:

$$J(\mathbf{x}, t) = det\,[\partial_{\mathbf{x}} \boldsymbol{\xi}] \tag{4.25}$$

Routine calculation using expression for the Jacobian, remembering $\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{u}(\boldsymbol{\xi}(\mathbf{x}, t), t)$ for fixed $\mathbf{x}$, shows that

$$\frac{\partial J}{\partial t} = [\nabla \cdot \mathbf{u}] J \tag{4.26}$$

Using $J(x, 0) = 1$, the above implies

$$J(x, t) = \exp\left[\int_0^t [\nabla \cdot \mathbf{u}](\phi(x, \tau), \tau) d\tau\right] \tag{4.27}$$

This relation implies in particular that unless $\nabla \cdot \mathbf{u}$ is singular, J *cannot* be zero for finite t.

Exercise: Prove relation (4.26)

An *incompressible* fluid is characterized by volume of W_t being fixed for any initial set W_0, *i.e.*

$$\frac{d}{dt} \int_{W_t} d\xi = 0 \tag{4.28}$$

This means

$$0 = \frac{d}{dt} \int_{W_0} J(\mathbf{x}, t) d\mathbf{x} = \int_{W_0} J_t d\mathbf{x} = \int_{W_0} J(\nabla \cdot \mathbf{u}) d\mathbf{x} \tag{4.29}$$

Since this is true for any W_0, *incompressibility* is equivalent to

$$\nabla \cdot \mathbf{u} = 0 \tag{4.30}$$

When incompressibility, *i.e.* (4.30) holds, the *continuity* equation (2.5) becomes:

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \tag{4.31}$$

while momentum equation (3.22) becomes

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta u + \rho \mathbf{b} \tag{4.32}$$

Equation (4.30), (4.31) and (4.32) constitute the complete equations, called the Navier-Stokes equation for *incompressible* fluids for unknown density, velocity and pressure functions ρ, $\mathbf{u}$ and p. Note that there are just as many scalar equations (five altogether) as the number of unknowns, which are three components of $\mathbf{u}$, along with scalars p and ρ.

5. **Incompressible uniform density Navier-Stokes equation**

Further simplifications occur when the initial fluid density $\rho(\mathbf{x},0)=\rho_0$, where ρ_0 does not depend on $\mathbf{x}$. In that case (4.31) clearly implies that $\rho(\mathbf{x},t)=\rho_0$ for all t. The momentum equation (4.32) simplifies to

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla\left(\frac{p}{\rho_0}\right)+\nu\Delta u+\mathbf{b}, \tag{5.33}$$

where $\nu\equiv\frac{\mu}{\rho_0}$ is a constant, called the *kinematic* viscosity, which has physical dimensions of $(Length)^2/Time$ In this limit, it is also common notation in the literature to replace $\frac{p}{\rho_0}$ by p. Equation (5.33) together with incompressibility equation:

$$\nabla\cdot\mathbf{u}=0 \tag{5.34}$$

constitutes four scalar equations for four scalar unknowns (three component of $\mathbf{u}$ and p.

Exercise: Suppose $\mathbf{x}=(x,y,z)$, $\mathbf{u}=(u,v,w)$. Express the *incompressible uniform density* Navier-Stokes equations (5.33) and (5.34) in terms of its scalar components.

5.1. **Initial Condition.**

Equations above have to be complimented by initial and boundary conditions. Appropriate initial condition is given by

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \tag{5.35}$$

where initial condition satisfies the incompressibility constraint $\nabla\cdot\mathbf{u}_0=0$. Initial conditions on pressure are inappropriate, since taking the divergence of (5.33) gives us the relation

$$-\Delta\frac{p}{\rho_0}=\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})+\nabla\cdot\mathbf{b} \tag{5.36}$$

Thus, pressure $p(\mathbf{x},t)$ is determined at each instant of time for known $\mathbf{u}(\mathbf{x},t)$ by inverting the Poisson equation (5.36), with boundary conditions discussed later. Initial conditions for p is therefore inappropriate in this limit.

5.2. **Boundary Condition for a solid boundary $\partial\Omega$.**

Equations and initial conditions have to be supplemented by boundary conditions. If Ω is finite, appropriate boundary condition on a solid boundary is the so-called *no-slip* boundary condition:

$$\mathbf{u}=\mathbf{v} \text{ on } \partial\Omega \tag{5.37}$$

where $\mathbf{v}$ is the specified velocity of boundary of the domain. For instance if Ω is a fluid container, whose boundary is in motion with velocity $\mathbf{v}$, specification of (5.37) implies that at on the boundary, the container particles and fluid particles share the same velocity; hence it is referred to as the *no-slip* boundary conditions. In the special case, when $\mathbf{v}=0$, *i.e.* the boundary is at rest, the no-slip boundary condition becomes $\mathbf{u}=0$ on $\partial\Omega$. Additionally, taking dot product of (5.33) with unit outwards normal $\mathbf{n}$ at the boundary $\partial\Omega$, we obtain for no-slip boundary condition,

$$\frac{\partial}{\partial n}\left(\frac{p}{\rho_0}\right)=-\mathbf{n}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}]\;+\;\nu\mathbf{n}\cdot(\Delta\mathbf{u})+\mathbf{n}\cdot\mathbf{b}, \tag{5.38}$$

which is a Neumann boundary condition for pressure p satisfying Poisson's equation (5.36) in Ω. Using $\nabla\cdot\mathbf{u}=0$, it is not difficult to show that the consistency condition

$$\int_{\partial\Omega}\frac{\partial p}{\partial n}dA=0$$

is satisfied by the right hand side of (5.38); and hence p is determined in terms of velocity $\mathbf{u}$ upto an additive constant.

If Ω is unbounded and extends to ∞, one needs to specify asymptotic conditions on $\mathbf{u}$ as $\mathbf{x} \to \infty$, either explicitly or implicity. For instance if we consider fluid moving past a fixed body, Ω is then the exterior of the body and an appropriate condition would be

$$\lim_{\mathbf{x}\to\infty} u(\mathbf{x}, t) = \mathbf{U}(t) \tag{5.39}$$

where $\mathbf{U}$ is velocity of the fluid at infinity.

5.3. **Boundary condition on a free-boundary.** The *no-slip* boundary condition (5.37) is not appropriate on any part of $\partial\Omega$, which is not a solid boundary. For instance, at a free-boundary like the surface of a drop/bubble or water-wave, we require instead that the **net** surface force at each point on the free boundary is zero. This follows from Newton's second law because the inertia of the free-surface is negligible[(7)].

Thus, if $\partial\Omega$ or parts of it is bounded on the other side by vacuum or some other fluid of negligible stress (air under most conditions), we replace (5.37) by

$$-p\mathbf{n} + \mathbf{T}\cdot\mathbf{n} = -\gamma\kappa\mathbf{n} \tag{5.40}$$

where γ is the surface tension coefficient and depends on the two fluids across the two sides of the free boundary and κ is the mean-curvature of the interface, which will be positive for a convex shape, such as shown in Fig. 1. For a *Newtonian incompressible* fluid, (5.40) becomes

$$-p\mathbf{n} + 2\mu\mathbf{S}\cdot\mathbf{n} = -\gamma\kappa\mathbf{n} \tag{5.41}$$

Boundary conditions become more involved when both fluids across $\partial\Omega$ have non-trivial stresses. However, the basic principle is still the same and involves a similar force balance at each point $\mathbf{x} \in \partial\Omega$.

5.4. **Dimensionless variable and Reynolds Number Re.** Consider the case of flow without external forcing[(8)] *i.e.* $\mathbf{b} = 0$. For incompressible constant density case, we can nondimensionalize all the variables of interest by introducing a length scale L and a velocity scale U. For instance in flow past a spherical body, we can take L to the the diameter of the sphere and U to be the fluid velocity magnitude at ∞. Without specifying precisely what they are, we think in general of L and U representing a typical length and velocity magnitude that characterizes the problem.

We introduce nondimensional variables:

$$\mathbf{u} = U\mathbf{u}' \ , \ t = \frac{L}{U}t' \ , \ \mathbf{x} = L\mathbf{x}' \ , \ p = \rho_0 U^2 p' \tag{5.42}$$

With this scaling, the *incompressible constant density* equations (5.33) and (3.22) become:

$$\partial_{t'}\mathbf{u}' + (\mathbf{u}'\cdot\nabla')\mathbf{u}' = -\nabla' p' + \frac{1}{Re}\Delta'\mathbf{u}' \tag{5.43}$$

$$\nabla'\cdot\mathbf{u}' = 0 \tag{5.44}$$

[(7)]Note $\partial\Omega$ has zero measure in $\mathbb{R}^3$, hence finite fluid density ρ implies zero mass

[(8)]The same analysis holds if $\mathbf{b} = \nabla\Psi_b$, since pressure p can be replaced by $p - \rho_0\Psi_b$, thus reducing it to $\mathbf{b} = 0$ case.

where

$$Re = \frac{UL}{\nu} \tag{5.45}$$

is a non-dimensional number, called the *Reynolds Number* that completely characterizes flows, when a free boundary is not involved.

In the mathematical literature, it is common to take the non-dimensional equations (5.43) and (5.44) (without the primes) as the incompressible constant density equations and think of Reynolds number given by (5.45) as the inverse of some non-dimensionalized viscosity ν.

The advantage of a non-dimensional form is that we can predict the flow past an aero-plane from wind-tunnel experiment on a laboratory object by ascertaining that Re number is the same. It is the same reason why a small object moving in water can generate the same flow conditions as a human being moving through honey, after appropriate rescaling. What matters is not the absolute size of the object, or typical velocity of motion or viscosity of fluid, but only the combination that goes into the definition of Re in (5.45). More precisely, from (5.43) and (5.44) it follows that in dimensional form,

$$\frac{\mathbf{u}}{U} = \mathbf{v}\left(\frac{\mathbf{x}}{L}, \frac{tU}{L}; Re\right)$$