Week 2 Notes, Math 8610, Tanveer

1. Incompressible constant density equations in different forms

Recall we derived the Navier-Stokes equation for incompressible constant density, *i.e.* homogeneous flows:

$$\mathbf{u}_t + [\mathbf{u} \cdot \nabla]\mathbf{u} = -\nabla p + \nu \Delta u + \mathbf{b} \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1.2}$$

where p is the scaled pressure (*i.e.* $\frac{p}{\rho_0}$ in previous notation). We note that the non-dimensional ν is simply $\frac{1}{Re}$, *i.e.* reciprocal of the Reynolds number.

There are alternate forms of Navier-Stokes equation besides (1.1)-(1.2). These are sometimes more valuable in analysis. If we apply the divergence operator to (1.1), which is well-defined if $u \in \mathbf{C}^2$, and use (1.2) to obtain

$$-\Delta p = \nabla \cdot ([\mathbf{u} \cdot \nabla]\mathbf{u}) + \nabla \cdot \mathbf{b} \tag{1.3}$$

Boundary condition for p on $\partial\Omega$ is found by taking the dot product of (1.1), interpreted as the limit of $x \to \partial\Omega$, with unit normal $\mathbf{n}$ on the boundary. If the boundary is at rest, no slip boundary condition implies $\mathbf{u} \cdot \mathbf{n} = 0$ and $[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{n} = 0$. Thus, we obtain

$$\frac{\partial p}{\partial n} = [\mathbf{b} + \nu \Delta \mathbf{u}] \cdot \mathbf{n} \text{ on } \partial\Omega \tag{1.4}$$

Given $\mathbf{u}$, the pressure p is determined by the linear elliptic PDE (1.3) with Neumann boundary condition (1.4). Note that the consistency condition for Neumann BC (1.4)

$$\int_\Omega \Delta p d\mathbf{x} = \int_{\partial\Omega} \frac{\partial p}{\partial n} dA$$

is indeed satisfied. Also, the solution p is only determined up to a time-dependent constant. However, this constant is irrelevant for determining velocity $\mathbf{u}$ since (1.1) involves only ∇p. Instead of using (1.1)-(1.2) to solve for $(\mathbf{u}, p)$, it is sometimes more convenient to use (1.1)-(1.3) to solve for $(\mathbf{u}, p)$. At each instant of time, given $\mathbf{u}$, (1.3) and (1.4) determine pressure p, which is then used in (1.1) to evolve $\mathbf{u}$ in time. The following Lemma is easily proved

Lemma 1.1. *Assume initial condition* $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ *is divergence free. Then* $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ *is a smooth solution to (1.1)-(1.2) if and only if it satisfies (1.1)-(1.3)*

Exercise: Prove Lemma 1.1.

There is yet another formulation of incompressible constant density Navier-Stokes involving vorticity, defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{1.5}$$

In component form, the i-th component of vorticity vector is:

$$\omega_i = \epsilon_{ijk} \partial_{x_j} u_k \equiv \epsilon_{ijk} u_{k,j} \tag{1.6}$$

where $u_{k,j}$ is a short-hand notation for $\partial_{x_j} u_k$ and ϵ_{ijk} are the components of the Levi-Civita tensor given by

$$\epsilon_{ijk} = 0 \text{ if any two indices equal }, \tag{1.7}$$
$$\epsilon_{ijk} = 1 \text{ if indices are distinct and even permutation of } \{123\} \tag{1.8}$$
$$\epsilon_{ijk} = -1 \text{ if indices distinct and odd permutation of } \{123\} \tag{1.9}$$

Note that

$$\omega_1 = u_{3,2} - u_{2,3} = \partial_{x_2} u_3 - \partial_{x_3} u_2$$
$$\omega_2 = u_{1,3} - u_{3,1} = \partial_{x_3} u_1 - \partial_{x_1} u_3$$
$$\omega_3 = u_{2,1} - u_{1,2} = \partial_{x_1} u_2 - \partial_{x_2} u_1$$

Another well-known property of the product of Levi-Civita Tensor will be useful later on.

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{1.10}$$

Expressing the displacement of a fluid particle in terms of a Taylor expansion in space relative to a reference point, it is not difficult to show that vorticity $\boldsymbol{\omega}$ is twice the the local rotation speed in a fluid. If we imagine a small light stick put in a fluid at $\mathbf{x}$ at time t, it will rotate with angular velocity $\frac{1}{2}|\boldsymbol{\omega}|$ with axis of rotation along $\boldsymbol{\omega}$.

We now apply the curl operator $\nabla\times$ to (1.1) we obtain

$$\boldsymbol{\omega}_t + \nabla \times [\mathbf{u} \cdot \nabla \mathbf{u}] = \nu \Delta \boldsymbol{\omega} + \nabla \times \mathbf{b} \tag{1.11}$$

Now, using vector identities we obtain

$$[\mathbf{u} \cdot \nabla]\mathbf{u} = \nabla \frac{\mathbf{u}^2}{2} - \mathbf{u} \times \boldsymbol{\omega} \tag{1.12}$$

So the i-th component of the second term on the left of (1.11) is:

$$\{\nabla \times [\mathbf{u} \cdot \nabla \mathbf{u}]\}_i = -\epsilon_{ijk}\partial_{x_j}\epsilon_{klm}\left[u_l \omega_m\right]$$
$$= [\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}]\left[u_{l,j}\omega_m + u_l \omega_{m,j}\right] = -u_{i,j}\omega_j + u_j \omega_{i,j} = [-(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}]_i$$

Therefore, (1.11) becomes:

$$\boldsymbol{\omega}_t + [\mathbf{u} \cdot \nabla]\boldsymbol{\omega} = [\boldsymbol{\omega} \cdot \nabla]\mathbf{u} + \nu \Delta \boldsymbol{\omega} + \nabla \times \mathbf{b} \tag{1.13}$$

Equation (1.13) combined with (1.2) and (1.5) determines $\boldsymbol{\omega}$ and $\mathbf{u}$. Note in this case, we have eliminated pressure p altogether.

Lemma 1.2. *Assume that initial condition $\mathbf{u}_0$ is smooth and satisfies $\nabla \cdot \mathbf{u}_0 = 0$. Then, $(u(\mathbf{x}, t), p(\mathbf{x}, t)$ is a smooth solution to (1.1)-(1.2) if and only if $(\boldsymbol{\omega}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t))$ is a solution satisfying (1.13)-(1.2) and (1.5).*

Proof. Clearly if $(\mathbf{u}, p)$ satisfies (1.1)-(1.2), then the above derivation shows that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ satisfies (1.13). (1.5) follows from definition of vorticity $\boldsymbol{\omega}$. Now we show the converse. Assume $(\boldsymbol{\omega}, \mathbf{u})$ satisfies (1.13),(1.5) and (1.2). Then, we know from the above process of derivation that (1.13) is the same as

$$\nabla \times [\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} - \mathbf{b}] = 0$$

Therefore, (1.1) follows since the curl of a vector field equals zero implies it is the gradient of a scalar, defined as pressure p. ∎

Using (1.5) and (1.2), it is possible to have a more explicit representation of $\mathbf{u}$ in terms of $\boldsymbol{\omega}$ in simple geometries. To see this, first notice that (1.2) implies

$$\mathbf{u} = \nabla \times \mathbf{A} \tag{1.14}$$

where $\mathbf{A}$ is generally called the vector potential. Now, (1.5), becomes

$$\nabla \times [\nabla \times \mathbf{A}] = \boldsymbol{\omega} \tag{1.15}$$

The following vector identity is not difficult to show $\nabla\times(\nabla\times\mathbf{A}) = -\Delta\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})$. Since the physical velocity $\mathbf{u} = \nabla\times\mathbf{A} = \nabla\times[\mathbf{A}+\nabla\phi]$, there is no loss of generality choosing $\nabla\cdot\mathbf{A} = 0$, as otherwise we can replace $\mathbf{A}$ by $\mathbf{A}+\nabla\phi$, where $-\Delta\phi = \nabla\cdot\mathbf{A}$. Hence

$$-\Delta\mathbf{A} = \boldsymbol{\omega} \tag{1.16}$$

In any geometry that allows an explicit Green's function for the $-\Delta$ operator, we can have an explicit integral representation for $\mathbf{A}$ and hence on taking the curl an explicit representation for $\mathbf{u}$ in terms of $\boldsymbol{\omega}$. For instance if $\Omega = \mathbb{R}^3$, then for ω decaying appropriately at ∞, inversion of (1.16) gives

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}' \tag{1.17}$$

Applying curl operator to (1.17), we obtain the so-called *Biot-Savart* relation for velocity $\mathbf{u}$ in terms of the vorticity $\boldsymbol{\omega}$:

$$\mathbf{u}(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\omega}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.18}$$

Note that in (1.18), we assume that the vorticity $\boldsymbol{\omega}(\mathbf{x},t)$ goes to zero sufficiently fast for the integral to make sense. Similar Biot-Savart relation can be derived for other simple geometries. Also, velocity $\mathbf{u}$ is assumed to go to zero at ∞ as well. We notice that (1.13) and (1.18) together can be thought of as an integro-differential equation for one-quantity $\boldsymbol{\omega}$.

2. **Ideal or Euler limit**

When Reynolds number $Re \to \infty$ or equivalently $\nu \to 0$ in (1.1)-(1.2), we formally get the so-called *Euler* equations for ideal incompressible uniform density fluid:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{b},\ \nabla\cdot\mathbf{u} \ = \ 0 \tag{2.19}$$

while we still use incompressibility relation (1.2). In the vorticity form, the Euler equation (2.19) takes the form

$$\boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nabla\times\mathbf{b} \tag{2.20}$$

Since the order of the equation is now reduced, it is no longer possible to satisfy the no-slip boundary condition $\mathbf{u} = \mathbf{v}$ on $\partial\Omega$ at a solid body. Instead, one requires equality of the normal components

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} \text{ on } \partial\Omega \tag{2.21}$$

The tangential components of fluid velocity need not be the same as the tangential velocity of the boundary; *i.e.* we allow for fluid to *slip* past the solid body. One way to understand how a Navier-Stokes solution in the limit of $\nu \to 0$ tends to an Euler solution is to imagine a thin boundary layer adjacent to the solid body. In the limit, this layer shrinks to zero size and corresponds to a singular distribution of vorticity on the solid boundary itself; this is called *a vortex sheet.*

On a free-boundary such as on the surface of a bubble/drop or water wave, we only consider balance of normal forces; tangential component is identically zero, *i.e.* we only use the condition

$$p = \sigma\kappa \text{ on } \partial\Omega \tag{2.22}$$

2.1. **Irrotational or Potential Flow.** In the vorticity form (2.20), it is clearly seen that if the solution to (2.20),(1.2) and (1.5) in $\Omega = \mathbb{R}^3$ is unique, then $\boldsymbol{\omega}(\mathbf{x}, 0) = 0$ implies $\boldsymbol{\omega}(\mathbf{x}, t) = 0$ for all $t > 0$, *i.e.* the flow is *irrotational* for all time. From (1.5), it follows that

$$\mathbf{u} = \nabla\Phi \tag{2.23}$$

where Φ is called the *velocity potential.* Then the divergence condition (1.2) implies

$$\Delta\Phi = 0 \tag{2.24}$$

Therefore in this case, we can solve Laplaces equation for the velocity potential and determine velocity through (2.23), without ever solving the Euler equation directly. Indeed, if the body force $\mathbf{b} = -\nabla V$ (in the case of gravity $V = gx_3$), we we may use $\boldsymbol{\omega} = 0$ and rewrite Euler equation (2.19) in the form

$$\nabla\left[\Phi_t + p + \frac{1}{2}\mathbf{u}^2 + V\right] = 0 \tag{2.25}$$

This gives rise to Bernoulli's equation[(1)]

$$\Phi_t + p + \frac{1}{2}\mathbf{u}^2 + V = 0 \tag{2.26}$$

Generally, we need an arbitrary function of time on the right side of (2.26); but this is not necessary since it can be absorbed in Φ, without affecting $\nabla\Phi$, which is the physical velocity. For steady flow, Bernoulli's law (2.26) becomes

$$p + \frac{1}{2}\mathbf{u}^2 + V = 0 \tag{2.27}$$

In the absence of a a body force, *i.e.* $V = 0$, we note that (2.27) implies that regions of large speed corresponds to small pressures, while in lower speed regions, pressure is higher. This is important in aero-dynamics.

2.1.1. *Potential Flow past a stationary sphere.* We now seek to determine the steady potential flow past a sphere of radius a with a constant uniform flow U at ∞ with no external forcing and determine the total total force exerted by fluid on the sphere.

We need to solve Laplace's equation outside a sphere. Consider a spherical coordinate system with origin at the center of the sphere. We take the x_3-axis along the direction of the flow at ∞, *i.e.* $\lim_{\mathbf{x}\to\infty} \mathbf{u}(\mathbf{x}) = U\hat{x}_3$, $\hat{x}_3$ being a unit vector in the direction of x_3. Since $\mathbf{u} = \nabla\Phi$. So,

$$\lim_{\mathbf{x}\to\infty} \left[\Phi(\mathbf{x}) - Ux_3\right] = 0$$

In spherical coordinates (ρ, θ, ϕ), this implies

$$\lim_{\rho\to\infty} \left\{\Phi(\rho, \theta, \phi) - U\rho\cos\theta\right\} = 0 \tag{2.28}$$

(1)Note p here is the scaled pressure, *i.e.* actual pressure/density.

Note that since equations boundary conditions and asymptotic conditions remain invariant with respect to rotation in ϕ. Therefore, the solution will not depend on ϕ. We seek a simple solution to $\Delta\Phi = 0$ in the form suggested by the θ dependence in (2.28):

$$\Phi(\rho, \theta) = f(\rho) \cos\theta \tag{2.29}$$

Using Laplace's equation representation in spherical coordinates, we have

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{2.30}$$

Substituting (2.29)

$$f'' + \frac{2}{\rho} f' - \frac{2}{\rho^2} f = 0 \tag{2.31}$$

This has solution

$$f(\rho) = C_1 \rho + C_2 \rho^{-2} \tag{2.32}$$

In order to match to the asymptotic condition (2.28), we have $C_1 = U$. Now, to satisfy the condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\rho = a$, we need $\partial_\rho \Phi(a, \theta) = 0$. this implies from (2.32) that

$$f'(a) = U - 2C_2 a^{-3} = 0 \text{ implying } C_2 = \frac{Ua^3}{2} \tag{2.33}$$

So, a solution for the potential flow problem is

$$\Phi(\rho, \phi) = U \cos\theta \left(\rho + \frac{a^3}{2\rho^2}\right) \tag{2.34}$$

Uniqueness of this solution follows within a large class of functions (for e.g. $H^1(\Omega)$ with continuity of derivative in $\overline{\Omega}$), using standard energy methods for Laplace's equation. The components of velocity $\mathbf{u}$ in spherical coordinates are

$$u_\rho = \partial_\rho \Phi = U \cos\theta \left(1 - \frac{a^3}{\rho^3}\right) \ , \ u_\theta = \frac{1}{\rho}\partial_\theta \Phi = -U \sin\theta \left(1 + \frac{a^3}{2\rho^3}\right) \tag{2.35}$$

From the expressions for components (u_ρ, u_θ) of velocity, we obtain on using steady Bernoulli equation, that scaled pressure

$$p(\rho, \theta) = -\frac{1}{2}\mathbf{u}^2 = -\frac{U^2}{2}\left\{\cos^2\theta \left(1 - \frac{a^3}{\rho^3}\right)^2 + \sin^2\theta \left(1 + \frac{a^3}{2\rho^3}\right)^2\right\} \tag{2.36}$$

On the sphere $\rho = a$, we obtain

$$p(a, \theta) = -\frac{9U^2}{8}\sin^2\theta \tag{2.37}$$

The force exerted by the fluid on the sphere is

$$\mathbf{F} = \int_{\rho = a} -p\mathbf{n} dA \tag{2.38}$$

Along the x_3 direction, since $n_3 = \cos\theta$, we have

$$F_3 = -2\pi a^2 \int_0^\pi \sin\theta \ \cos\theta p(a, \theta) d\theta = 0 \tag{2.39}$$

Since $n_1 = \sin\theta\cos\phi$ and $n_2 = \sin\theta\sin\phi$, it follows that

$$F_1 = -a^2 \int_0^{\pi} \left\{ \int_0^{2\pi} \sin\theta p(a,\theta)\sin\theta \ \cos\phi d\phi \right\} d\theta \tag{2.40}$$

So, there is no force on the sphere! This is called the D'Alembert paradox and can be shown to be generally true for steady potential flow past any body.

The reason there is no force is because we have (i) assumed that there is no viscosity, (ii) assumed that vorticity, which is generated at the boundary in a singular way because of the no-slip boundary condition is somehow confined to the boundary. This is not the case for a realistic flow field.

Further notice from (2.35) that while the radial velocity component $u_\rho = 0$ at $\rho = a$, this is not true for the θ component; *i.e.* fluid does *slip* past the cylinder at $\rho = a$. We *cannot* generally impose a physical no-slip boundary condition for Euler equation.

Exercise: Determine potential flow past an infinite stationary cylinder of radius a. You can assume the axis of the cylinder to be aligned along the x_3-axis and that the flow has no x_3 dependence. You can also assume that $\mathbf{u} - U\hat{x}_1 = o(1/\sqrt{x_1^2 + x_2^2})$ as $(x_1, x_2) \to \infty$.

Exercise: For a general smooth shaped body, use Bernoulli's equation and divergence theorem to show that the steady potential flow past a stationary body exerts no force. What assumptions on $\mathbf{u}$ are necessary at ∞ for the above to be true?

2.1.2. *Two dimensional Potential Flow.* For two dimensional flow, *i.e.* no dependence on x_3, such as flow past an infinite cylinder, we may as well consider the reduced dimensional problem where the independent space variable $\mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$.

In two dimensions since every solution to $\Delta\phi = 0$ is associated with an analytic function W of $z = x_1 + ix_2$[(2)] so that

$$\Phi(x_1, x_2) = \mathrm{Re}W\ (x_1 + ix_2) = \ \mathrm{Re}W(z) \tag{2.41}$$

$W(z)$ is usually referred to as the *complex velocity potential.* It's imaginary part Ψ is called the *stream function.* From use of Cauchy Riemann conditions and $\mathbf{u} = \nabla\Phi$, it follows that

$$\partial_{x_1}\Phi = \partial_{x_2}\Psi = u_1\ ,\ \partial_{x_2}\Phi = -\partial_{x_1}\Psi = u_2 \tag{2.42}$$

Stream function has also a nice physical meaning–the fluid flow is always tangent to *level set* of Ψ determined by $\Psi(x_1, x_2) =$ Constant. To show this we note if (dx_1, dx_2) is parallel to (u_1, u_2), then $u_1 dx_2 - u_2 dx_1 = 0 = \partial_{x_2}\Psi dx_2 + \partial_{x_1}\Psi dx_1 = d\Psi$. The level sets of the stream function are called *streamlines.*

Also, the total fluid flux (in 2-D sense) between two *streamlines* $\Psi = c$ and $\Psi = d$ is simply $d - c$. To see this take two points on the two steamlines and draw a smooth line $(x_1(s), x_2(s))$, $s \in [0,1]$ joining them. The flux through this line, whose normal is clearly $(x_2', -x_1')$, is given by

$$\begin{aligned} \mathit{Flux} = \int_0^1 [u_1 x_2' - u_2 x_1'] ds &= \int_0^1 \frac{d}{ds}\Psi(x_1(s), x_2(s)) ds \\ &= \Psi(x_1(1), x_2(1)) - \Psi(x_1(0), x_2(0)) = d - c \end{aligned}$$

[(2)]There is no assumption on fluid being steady. W can also be a function of time t, but this dependence is suppressed here.

The condition on Euler flows that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for a stationary boundary implies fluid flow is tangential to the boundary. For a 2-D potential flow, this implies that the free boundary is a *stream line* and we can set the condition

$$\Psi(x, y) = c \tag{2.43}$$

In terms of the complex velocity potential $W(z)$, we note that

$$\frac{dW}{dz} = \partial_{x_1}\Phi + i\partial_{x_1}\Psi = \partial_{x_1}\Phi - i\partial_{x_2}\Phi = u_1 - iu_2 \tag{2.44}$$

Because of the relation (2.44) with velocity (u_1, u_2), $\frac{dW}{dz}$ is usually called the *complex velocity.*

So, the problem of determining potential flow in 2-D reduces to finding an appropriate analytic function $W(z)$ which satisfies certain boundary conditions on $\partial\Omega$ and at ∞ (if $\infty \in \Omega$). Sometimes, it is appropriate to look for solutions where $W(z)$ can have singularities at one or more points. For instance, we can consider an idealized *point source* at $z = z_0$, which is characterized by fluid being added to the domain Ω, say at a rate m. This would correspond to specifying

$$W(z) \sim \frac{m}{2\pi}\log(z - z_0) + O(1) \text{ as z} \to \text{z}_0 \tag{2.45}$$

To show that this corresponds to a source of spewing out fluid at $z = z_0$ (corresponding to $\mathbf{x} = \mathbf{x}_0$) at a rate m consider the boundary contribution of the 2-D flux[(3)]

$$2 - \text{D flux} = \int_{|\mathbf{x}-\mathbf{x}_0|=\epsilon} \mathbf{u} \cdot \mathbf{n} ds = \epsilon\int_0^{2\pi} \partial_r\Phi d\theta \ , \text{ where } z - z_0 = re^{i\theta} \tag{2.46}$$

We now note that $\partial_r\Phi d\theta = \text{Re}\left\{\partial_r W \frac{dz}{i(z-z_0)}\right\}$. Further

$$\frac{\epsilon}{z - z_0}\partial_r W = e^{-i\theta}\partial_r W = \frac{dW}{dz}$$

So, from (2.46), the 2-D flux around $z = z_0$ is given by

$$\int_{|\mathbf{x}-\mathbf{x}_0|=\epsilon} \mathbf{u} \cdot \mathbf{n} ds = \text{Re}\left\{-i\oint_{|z-z_0|=\epsilon} \frac{dW}{dz}dz\right\} = m \tag{2.47}$$

from contour integration. Multiple sources at different points correspond to log singularities of $W(z)$ at such points.

Yet, another type of singularity arises in idealized potential flow. These are called *point vortices.* A point vortex $\mathbf{x} = \mathbf{x}_0$ of *strength* Γ is a singularity of $\mathbf{u}$ for which

$$\oint_{|\mathbf{x}-\mathbf{x}_0|=\epsilon} \mathbf{u} \cdot \tau ds = \Gamma \ , \text{ where } \tau \text{ is the unit tangent vector} \tag{2.48}$$

for any $\epsilon > 0$ sufficiently small, where the integral is traversed in the positive sense (anti-clockwise). A point vortex corresponds to a singularity of $W(z)$ at the corresponding complex point $z_0 = x_{0,1} + ix_{0,2}$, such that

$$W(z) \sim \frac{-i\Gamma}{2\pi}\log(z - z_0) + o(1) \tag{2.49}$$

(3) Note this involves arc-length integrals instead of area integral in 3-D

Exercise: Show that if a 2-D potential flow $\mathbf{u}$ has a point vortex singularity at $\mathbf{x}_0$, then the corresponding complex potential $W(z)$ satisfies (2.49).

Because of the connection to complex analytic functions, and powerful complex variable methods including conformal map, many problems of 2-D potential flow can be solved explicitly, both for steady and time-dependent problems. We give some simple examples in the subsections below.

2.1.3. *Steady Flow past a flat plate aligned parallel to flow.* Consider the steady flow past a flat plate geometry, where the plate location is given by

$$\partial\Omega = \{(x_1, x_2) : -1 < x_1 < 1 \ , \ x_2 = 0\}$$

We seek solution so that as $\mathbf{x} \to \infty$,

$$\mathbf{u} - U\hat{x}_1 = o(1/r) \text{ as} r = \sqrt{x_1^2 + x_2^2} \to \infty \tag{2.50}$$

Using complex velocity $\frac{dW}{dz}$, the condition (2.50) becomes

$$\frac{dW}{dz} = U + o(1/z) \text{ as } z \to \infty \tag{2.51}$$

So,

$$W(z) = Uz + o(1) \text{ as } z \to \infty \tag{2.52}$$

We just notice that $W(z) = Uz$ satisfies the streamline condition on $\partial\Omega$ since

$$\Psi = \text{Im} W(x_1 + i0) = \text{Im} U x_1 = 0 \tag{2.53}$$

Therefore, the solution in this case is trivial

$$W(z) = Uz \text{ , implying } \frac{dW}{dz} = U \text{ i.e. } (u_1, u_2) = (U, 0) \tag{2.54}$$

So, we have a uniform flow and the presence of the plate makes no difference. This is expected since the flow is parallel to the plate and the plate has no effect on the flow.

2.1.4. *Steady Flow Past a Stationary Cylinder.* We want flow past a cylinder. Consider radius of the cylinder to be of radius 1, as it simplifies algebra a bit. However, the same method is applicable for any radius. As $z \to \infty$, we require as before for a flat plate, that the asymptotic condition (2.52) is satisfied.

However, we note that the analytic function

$$\zeta = f(z) = \frac{1}{2}(z + 1/z) \tag{2.55}$$

maps the domain exterior of the unit circle to the domain exterior of the domain of the slit joining $\zeta = -1$ to $\zeta = +1$. Its inversion is given by

$$z = f^{-1}(\zeta) = \zeta + \sqrt{\zeta^2 - 1} \tag{2.56}$$

Further, the mapping function (2.55) implies that as $\zeta \to \infty$, $\zeta \sim \frac{z}{2}$; hence the asymptotic condition (2.52) on $W(z)$ becomes

$$W(z(\zeta)) \sim 2U\zeta + o(1/\zeta) \text{ as } \zeta \to \infty \tag{2.57}$$

We know from previous section that

$$W(z(\zeta)) = 2U\zeta \tag{2.58}$$

satisfies the streamline condition on the real axis segment $(-1, 1)$ in the ζ-plane. But using (2.55), it follows that

$$W(z) = U\left(z + \frac{1}{z}\right)\ , \text{ implying } \Phi(r,\theta) = U\left(r + \frac{1}{r}\right)\cos\theta \tag{2.59}$$

is the solution we seek. We can check directly that on $|z| = 1$, W is real and so the boundary condition $\Psi = \mathrm{Im}W = 0$ is satisfied.