Week 3 Notes, Math 865, Tanveer

1. More on 2-D potential flow solutions

1.0.1. *Circle Theorem for 2-D potential flow.* For 2-D flow, it is sometimes useful to note that if flow is known in the absence of any boundary, then it possible to determine the flow in the presence of a stationary circle. This is because of the following circle Theorem.

Theorem 1.1. *(Circle Theorem) If $W_f(x_1 + ix_2)$ is the complex velocity potential for $\Omega = \mathbb{R}^2$ satisfying given conditions at ∞ and at possible singularities (specified sources and sinks for instance) located in $|x_1 + ix_2| > a > 0$, then*

$$W(z) = W_f(z) + \left\{ (W_f(a^2/z^*) \right\}^* \tag{1.1}$$

is the complex velocity potential for the domain $\Omega = \left\{ (x_1, x_2) : x_1^2 + x_2^2 > a^2 \right\}$

Proof. Since conditions at ∞ and singular conditions at finite points outside $|z| > a$ are satisfied by $W_f(z)$, it is only necessary to check that $[W_f(a^2/z^*)]^*$ is analytic, singularity free in $|z| \geq a$, finite at $z = \infty$ and that (1.1) satisfies Im $W = 0$ on $|z| = a$. The latter condition is checked directly by substituting $z = ae^{i\theta}$ and noting that the resulting expression in (1.1) is purely real. It also seen that the process of taking two complex conjugation in $[W_f(a^2/z^*)]^*$ results in an analytic function. Also, since $|z| > a$ is mapped to $|z| < a$ by the mapping a^2/z^*, the only singularity of this function is inside the unit circle and not within the domain of interest. Further, since $W_f(0)$ is finite, it follows that $[W_f(a^2/z^*)]^*$ is finite as $z \to \infty$. Thus, $W(z)$ is indeed the vector potential corresponding to the potential flow outside the circle of radius a. ∎

Example: Consider the problem of finding flow past a circular solid body when $\mathbf{u} \sim U\hat{x}^1$ as $\mathbf{x} = (x_1, x_2) \to \infty$. Assume we also have a vortex at $(x_1, x_2) = (b, c)$ with circulation κ where $b^2 + c^2 > 1$. We want to determine the flow.

In the absence of any body, we have from given condition

$$W_f(z) \sim Uz \text{ as } z \to \infty$$

Near $z = z_0 = b + ic$, the condition on $W_f(z)$ becomes

$$W_f(z) \sim \frac{-i\kappa}{2\pi} \log(z - z_0)$$

Therefore, in the absence of any body, we must have

$$W_f(z) = Uz - \frac{i\kappa}{2\pi} \log(z - z_0)$$

Therefore, from circle theorem:

$$W(z) = Uz - \frac{i\kappa}{2\pi} \log(z - z_0) + U/z + \frac{i\kappa}{2\pi} log(1/z - z_0^*)$$

Remark 1.1. *Circle Theorem together with conformal map allows one to explicitly calculate potential flows past many geometries of interest.*

2. Euler Flow with Vorticity in $\Omega \subset \mathbb{R}^3$: Kelvin's Circulation Theorem

As mentioned earlier, potential flows are very special. Even when viscosity effects are small, generally, we do expect vorticity $\boldsymbol{\omega}$ to be important. An important property of vorticity is given by Kelvin's circulation theorem.

Theorem 2.1. *Kelvin's circulation Theorem Let $\mathbf{u}(\mathbf{x}, t)$ is a smooth solution to the Euler equation, with forcing $\mathbf{b} = -\nabla V$. Let $C(t)$ be a closed curve in Ω that moves with the fluid, i.e. $C(t)$ is the image of the map of some initially closed curve C_0 under the flow: $\frac{d}{dt}\mathbf{x} = \mathbf{u}(\mathbf{x}(t), t)$. Then, the circulation Γ around the curve $C(t)$, is preserved in time, i.e.*

$$\frac{d}{dt}\Gamma_{C(t)} = 0 \ , \text{ where } \Gamma_{C(t)} = \oint_{C(t)} \mathbf{u} \cdot \mathbf{ds}$$

Proof. Let $\mathbf{x} = \mathbf{X}(\boldsymbol{\xi}, t)$ denote the map generated by

$$\frac{d}{dt}\mathbf{x} = \mathbf{u}(\mathbf{x}(t), t), \ \mathbf{x}(0) = \boldsymbol{\xi}$$

Let $\boldsymbol{\xi} = \boldsymbol{\Xi}(\sigma)$, $\sigma \in [0, 1]$ denote the parametrized curve C_0 in the anti-clockwise sense. Then,

$$\Gamma_{C(t)} = \oint_{C(t)} \mathbf{u} \cdot \mathbf{ds} = \int_{\sigma=0}^{1} \mathbf{u}(\mathbf{X}(\boldsymbol{\xi}, t), t) \cdot [\mathbf{X}_{\boldsymbol{\xi}} \boldsymbol{\Xi}_{\sigma}] \, d\sigma$$

Then,

$$\begin{aligned} \frac{d}{dt}\Gamma_{C(t)} &= \int_0^1 \left\{ \frac{D}{Dt}\mathbf{u} \cdot [\mathbf{X}_{\boldsymbol{\xi}} \boldsymbol{\Xi}_{\sigma}] + \mathbf{u} \cdot [\mathbf{u}_{\boldsymbol{\xi}} \boldsymbol{\Xi}_{\sigma}] \right\} d\sigma \\ &= -\oint_{C_t} \nabla(p + V) \cdot \mathbf{ds} + \int_0^1 \frac{d}{d\sigma}\frac{1}{2}\mathbf{u}^2(\mathbf{X}(\boldsymbol{\Xi}(\sigma), t), t) d\sigma = 0, \end{aligned} \tag{2.2}$$

since $\nabla(p + V) \cdot \mathbf{ds} = d[p + V]$ on the curve. ∎

Using the above and Stokes theorem: $\oint_C \mathbf{u} \cdot \mathbf{ds} = \int_{\boldsymbol{\Sigma}} \boldsymbol{\omega} \cdot \mathbf{n} d\mathbf{x}$ on a surface with boundary C, where $\mathbf{n}$ is normal to the surface, we obtain the following result:

Corollary 2.1. *Helmholtz Law of Vorticity Conservation: For a smooth solution to Euler equation, with $\mathbf{b} = -\nabla V$, the vorticity Flux $\mathcal{F}_{\Sigma(t)}$ through a surface $\Sigma(t)$ moving with the fluid:*

$$\mathcal{F}_{\Sigma(t)} = \int_{\Sigma(t)} \boldsymbol{\omega} \cdot \mathbf{n} d\mathbf{x}$$

is constant in time.

Remark 2.2. *Thus, we note that if the area of $\Sigma(t)$ decreases in time, the magnitude of $\boldsymbol{\omega}$ must increase. So, in the absence of external torque, if a rotating fluid is brought closer to the center of the rotation, it speeds up. This is similar to an ice-dancer rotating faster when the person brings his/her arms closer to the body.*

Remark 2.3. *The presence of viscosity ν does not preserve circulation since, we can show that for viscous fluid*

$$\frac{d}{dt}\Gamma_{C(t)} = \nu \oint_{C(t)} (\Delta \mathbf{u}) \cdot \mathbf{ds}$$

Exercise: Show the above equality.

3. 2-D vortex dynamics: stream function vorticity formulation

Consider now the special case of 2-D Euler flow with vorticity. In the context of 3-D flow, the 2-D reduction corresponds to $\mathbf{u} = (u_1, u_2, 0)$ and pressure p not depending at all on x_3. In this case, calculation shows that vorticity

$$\boldsymbol{\omega} = (0, 0, \omega) = (0, 0, \partial_{x_1} u_2 - \partial_{x_2} u_1) \tag{3.3}$$

Thus, vorticity is characterized by the scalar ω; it is common to call ω itself as the 2-D vorticity. The expression (3.3) also implies that the term

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0 \tag{3.4}$$

since $\mathbf{u}$ does not depend on x_3, which is the direction of $\boldsymbol{\omega}$. With this observation, we have from the vorticity equation of the Euler equation the following reduction in 2-D

$$\omega_t + \mathbf{u} \cdot \nabla \omega = [\nabla \times \mathbf{b}]_3 = \partial_{x_1} b_2 - \partial_{x_2} b_1 \equiv f \tag{3.5}$$

If $f = 0$, the above reduces to

$$\frac{D\omega}{Dt} = 0, \text{ implying } \omega(\mathbf{x}(\boldsymbol{\xi}, t), t) = \omega(\boldsymbol{\xi}, 0) = \omega_0(\boldsymbol{\xi}), \tag{3.6}$$

which means that the vorticity is constant on a fluid particle even as it moves around in time. This is unlike 3-D, where we can show that

$$\boldsymbol{\omega}(\mathbf{x}(\boldsymbol{\xi}, t), t) = \mathbf{X}_{\boldsymbol{\xi}} \ \boldsymbol{\omega}_0(\boldsymbol{\xi}) \tag{3.7}$$

Here $\mathbf{X}_{\boldsymbol{\xi}}(\mathbf{x}, t)$ depends on time and and causes vorticity to be amplified or reduced. We will see an explicit example of vortex stretching later.
Exercise: Prove the relation (3.7).

We return to more discussion of 2-D flows. First, we had from before that $\nabla \cdot \mathbf{u} = 0$

$$\mathbf{u} = \nabla \times \mathbf{A} \tag{3.8}$$

Since $u_3 = 0$, and $\mathbf{u}$ only depends on (x_1, x_2), there is no loss of generality in choosing

$$\mathbf{A} = (0, 0, \Psi) \tag{3.9}$$

and (3.8) implies

$$u_1 = \partial_{x_2} \Psi \ , \ u_2 = -\partial_{x_1} \Psi \tag{3.10}$$

So, using (3.3), we obtain

$$\omega = -\left[\partial_{x_1}^2 \Psi + \partial_{x_2}^2 \Psi\right] = -\Delta \Psi \tag{3.11}$$

With relation (3.10), we can actually write (3.5) in the form

$$\omega_t + \Psi_{x_2} \omega_{x_1} - \Psi_{x_1} \omega_{x_2} = f \tag{3.12}$$

For steady flow, with $f = 0$, the above reduces to

$$\omega = F(\Psi) \tag{3.13}$$

for essentially arbitrary differentiable function F and then (3.11) becomes a nonlinear equation for streamfunction

$$-\Delta \Psi = F(\Psi) \tag{3.14}$$

Remark 3.1. *There exists a general class of exact solutions for a special choice of $F(\Psi) = e^{2\Psi}$, and for a few other choices of Ψ. Some of these are expressible in complex variable formulation. However, without considering the effect of viscosity, there is no way to determine which $F(\Psi)$ is relevant.*

In general, for unsteady flow, (3.13) is not valid. In that case, equations (3.11) and (3.12) have to be satisfied simultaneously. It is sometimes convenient to use the appropriate Green's function $G(\mathbf{x}, \mathbf{x}')$ of $-\Delta$ for the Dirichlet problem (because boundaries $\partial\Omega$ are streamlines) to invert the relation (3.11). If $\Omega = \mathbb{R}^2$, then

$$\Psi(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(\mathbf{x}', t) \log |\mathbf{x} - \mathbf{x}'| \; d\mathbf{x}' \tag{3.15}$$

Velocities are given by

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^2} \omega(\mathbf{x}', t) K(\mathbf{x}, \mathbf{x}') \tag{3.16}$$

where the Kernel

$$K(\mathbf{x}, \mathbf{x}') = (\partial_{x_2}, -\partial_{x_1})\, G(\mathbf{x}, \mathbf{x}') \tag{3.17}$$

Equation (3.16) is the 2-D version of *Biot-Savart* law.

4. Exact radial vorticity solution, with and without viscosity

A simple exact solution is possible when the initial vorticity distribution is radial, *i.e.*

$$\omega(\mathbf{x}, 0) = \omega_0(|\mathbf{x}|)$$

and $f = 0$ in (3.12). In that case, the solution we seek is $\omega = \omega(r, t)$, where $r = \sqrt{x_1^2 + x_2^2} = |\mathbf{x}|$. Because of radial distribution of vorticity, it is clear that we can seek stream function Ψ satisfying (3.11) also has the form $\Psi = \Psi(r, t)$. Then, velocity in the polar coordinates

$$\mathbf{u} = (u_r, u_\theta) = \left(\frac{1}{r} \partial_\theta \Psi, -\partial_r \Psi \right) = (0, -\partial_r \Psi) \tag{4.18}$$

So, the velocity $\mathbf{u}$ is orthogonal to $\nabla\omega$, the latter being directed radially. Thus, (3.12) reduces simply to:

$$\omega_t = 0 \text{ implying } \omega(r, t) = \omega_0(r) \tag{4.19}$$

meaning that there is no-time dependence. Solving for Ψ we obtain

$$-\Psi_{rr} - \frac{1}{r} \Psi_r = \omega_0(r) \tag{4.20}$$

Therefore,

$$-\left[r \Psi'(r) \right]' = r \omega_0(r) \ , \text{ implying } v_\theta = -\Psi'(r) = \frac{1}{r} \int_0^r s \omega_0(s) ds \tag{4.21}$$

Therefore, in cartesian coordinates,

$$\mathbf{u} = \frac{1}{|\mathbf{x}|^2} (-x_2, x_1) \int_0^r s \omega_0(s) ds \tag{4.22}$$

In this problem because of the alignment of gradient of vorticity with respect induced velocity, vorticity is not affected by the velocity at all. This is a rather exceptional situation. Indeed, if we included viscosity, the only change would be that (4.19) would be replaced

$$\omega_t = \nu \Delta \omega \ , \text{ with } \omega(\mathbf{x}, 0) = \omega_0(|\mathbf{x}|) \tag{4.23}$$

Since 2-D heat equation can be readily solved in terms of the Heat Kernel, we have

$$\omega(\mathbf{x}, t) = \frac{1}{4\pi\nu t} \int_{\mathbb{R}^2} \exp\left[-\frac{|\mathbf{x} - \mathbf{x}'|^2}{4\nu t} \right] \omega_0(|\mathbf{x}'|) d\mathbf{x}' \tag{4.24}$$

It is not difficult to argue from (4.24) that ω only depends on $\mathbf{x}$ through $|\mathbf{x}|$, *i.e.* the above is a radial solution. The relation (4.22) is affected only slightly in that in this case, with viscosity, we have

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{|\mathbf{x}|^2} (-x_2, x_1) \int_0^r s\omega(s, t) ds, \tag{4.25}$$

So, the fluid velocity $\mathbf{u}$ is again only in the θ-direction. We note that with viscosity, the vorticity diffuses outwards and as $t \to \infty$, vorticity tends to zero. Consequenty, $\mathbf{v}$ given by (4.25) also goes to zero at $t \to \infty$.

4.1. **Explicit Example of Vortex Stretching in 3-D: Burger's vortex.** Consider a 3-D flow with a 2-D radial vorticity field, in the form,

$$\mathbf{u}(\mathbf{x}, t) = (-\frac{\alpha}{2} x_1, -\frac{\alpha}{2} x_2, \alpha x_3) + \frac{1}{\sqrt{x_1^2 + x_2^2}} (-x_2, x_1, 0) \int_0^r s\omega(s, t) ds, \text{ where } r = \sqrt{x_1^2 + x_2^2} \tag{4.26}$$

Notice that on taking the curl operation, we have

$$\nabla \times \mathbf{u} = (0, 0, \omega) \tag{4.27}$$

Indeed, the vorticity field depends only on r and t and the direction of vorticity is $\hat{x}_3$. In this case, since the flow-field has a linear x_3 component, $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 2\alpha\omega$. Therefore, the vorticity equation becomes

$$\omega_t - \frac{\alpha}{2} x_1 \partial_{x_1} \omega - \frac{\alpha}{2} \alpha x_2 \partial_{x_2} \omega + \alpha \omega = \nu \Delta \omega \ , \ \omega(\mathbf{x}, 0) = \omega_0(r) \tag{4.28}$$

A change in dependent and independent variables

$$\tau = \frac{\nu}{\alpha} \left(e^{\alpha t} - 1 \right) \ , \ \mathbf{y} = e^{\alpha t/2} \mathbf{x} \ , \ \tilde{\omega} = e^{-\alpha t} \omega \tag{4.29}$$

results in

$$\hat{\omega}_\tau = \nu \Delta \tilde{\omega} \ , \ \tilde{\omega}(\mathbf{y}, 0) = \omega_0(|\mathbf{y}|). \tag{4.30}$$

So the solution is given by

$$\hat{\omega}(\mathbf{y}, \tau) = \frac{1}{4\pi\nu\tau} \int_{\mathbb{R}^2} \exp\left[-\frac{|\mathbf{y} - \mathbf{y}'|^2}{4\nu\tau} \right] \omega_0(|\mathbf{y}'|) d\mathbf{y}' \tag{4.31}$$

So,

$$\omega(\mathbf{x}, t) = e^{\alpha t} \int_{\mathbb{R}^2} H\left(\mathbf{x} e^{\alpha t/2} - \mathbf{y}', \frac{\nu}{\alpha} \left[e^{\alpha t} - 1 \right] \right) \omega_0(|\mathbf{y}'|) d\mathbf{y}' \ , \text{ where } H(\mathbf{x}, t) = \frac{1}{4\pi t} \exp\left[-\frac{|\mathbf{x}|^2}{4t} \right] \tag{4.32}$$

Exercise: Show that as $t \to \infty$,

$$\omega(\mathbf{x}, t) \sim \frac{\alpha}{2\nu} \exp\left[-\frac{\nu r^2}{4\nu} \right] \int_0^\infty s\omega_0(s) ds \tag{4.33}$$

Burger's solution above shows the effect of straining flow on a single vortex column directed along the x_3-axis. The *straining flow* defined by the linear part of $\mathbf{u}$, is given by $(-\alpha x_1, -\alpha x_2, \alpha x_3)$. This is irrotational and divergence free. Nonetheless, this straining flow squeezes the initial vorticity towards $r = 0$ and tries to intensify it, while viscosity tries to spread it through diffusion. The steady state reached at ∞ in the exercise above shows the balance between these two effects.