Week 4 Notes, Math 8610, Tanveer

1. More Exact Solutions with nonzero Vorticity

1.1. Flow in a pipe: Steady axisymmetric Poiseuille Flow. Consider the steady unidirectional flow in an infinitely long pipe of radius a. To solve this problem, it is convenient to consider the Navier-Stokes equation in the cylindrical coordinates (r, θ, z) in the absence of any external force ($\mathbf{b} = 0$). In this coordinate system $\mathbf{u} = (u_r, u_\theta, u_z)$. The scalar components of the Navier-Stokes equation are given by[(1)]:

$$\partial_t u_z + (\mathbf{u} \cdot \nabla) u_z = -\partial_z p + \nu \Delta u_z \tag{1.1}$$

$$\partial_t u_r + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\phi^2}{r} = -\partial_r p + \nu \left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \partial_\theta u_\theta \right) \tag{1.2}$$

$$\partial_t u_\theta + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{r} \partial_\theta p + \nu \left(\Delta u_\theta + \frac{2}{r^2} \partial_\theta u_r - \frac{u_\theta}{r^2} \right) \tag{1.3}$$

$$\partial_r u_r + \frac{1}{r} \partial_\theta u_\theta + \partial_z u_z + \frac{1}{r} u_r = 0 \tag{1.4}$$

where in cylindrical coordinate system

$$\mathbf{u} \cdot \nabla = u_r \partial_r + u_\theta \frac{1}{r} \partial_\theta + u_z \partial_z \tag{1.5}$$

$$\Delta V = \partial_r^2 V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial_\theta^2 V + \partial_z^2 V \tag{1.6}$$

For a steady pipe-flow, we orient the pipe axis along the z-axis. We are looking for a simple uni-directional solution $\mathbf{u}(\mathbf{x}, t) = (0, 0, u_z(r))$, with $p = p(r, z)$. Then, from inspection, it is clear that the continuity equation (1.4) is immediately satisfied. Also, on inspection, the r and θ component of the momentum equations, given in (1.2) and (1.3), are also satisfied provided $\partial_r p = 0$. This implies $p = p(z)$. In equation (1.1), we note $\mathbf{u} \cdot \nabla u_z = 0$, since $u_r = 0$ and u_z only depends on r. So, this equation reduces to:

$$0 = -\partial_z p + \nu \left(\partial_r^2 u_z + \frac{1}{r} \partial_r u_z \right) \tag{1.7}$$

Since u_z can only depend on r, $\partial_z p$ cannot depend on r. Earlier, we argued that the pressure p does not depend on r. So

$$-\partial_z p = G = \text{constant} \tag{1.8}$$

[(1)]Note: p is not the actual pressure, but the scaled pressure *pressure/density*

From (1.7) on multiplying the equation by r/ν, we obtain

$$ru_z'' + u_z' = \frac{G}{\nu} r \tag{1.9}$$

where $'$ denotes derivative with respect to r. Since the left side of (1.9), is $(ru_z')'$, on integration, we obtain a general solution in the form:

$$u_z(r) = \frac{G}{4\nu}\left(-r^2 + C_1 \log r + C_2\right) \tag{1.10}$$

The solution $u_z(r)$ is singular at $r = 0$ if $C_1 \neq 0$ and hence does not conform to a physically acceptable solution, unless $C_1 = 0$[(2)]

Further $C_2 = 1$ in order that the no-slip condition $\mathbf{u} = 0$ is satisfied at the walls $r = a$. Note that in this case, since only one component of velocity is nonzero, this corresponds in this case to $u_z(a) = 0$. Therefore, flow in a pipe is cylindrical coordinate system by

$$\mathbf{u} = \left(0, 0, \frac{G}{4\nu}[a^2 - r^2]\right) \tag{1.11}$$

where (1.8) determines pressure. Note that the velocity in (1.11) is given by a parabolic profile with maximum velocity at $r = 0$, at the center of the pipe.

There is nothing to determine scaled pressure gradient G; this is something that is specified. This is related to the flow rate Q (volume of fluid going through any section of the pipe per second) since

$$Q = \int_0^a u_z(r) 2\pi r dr = \frac{\pi G a^4}{8\nu} \tag{1.12}$$

We note that for a given pressure gradient G, the flow rate Q scales inversely with viscosity ν. The greater the viscosity, smaller the flow rate; this is physically sensible since friction is greater when viscosity is larger. Further, according to (1.12), the flow rate Q scales like the the fourth power of the radius. This is because the area grows like a^2, where as the average velocity across the pipe determined from (1.11) also grows like a^2.

Also, note that if the pipe length l is not infinite but large, then

$$G \approx \frac{p_0 - p_1}{L}\ , \tag{1.13}$$

where p_0 and p_1 are the pressures at the left and right end of the pipes, say at $z = -L/2$ and $z = L/2$ respectively.

We note that instead of a pressure gradient, the flow may also be driven by gravity. For instance, for a vertically aligned pipe, the role

(2) $C_1 \neq 0$ corresponds physically to a finite force per-unit-length exerted at $r = 0$. There is no such forcing at the center of a pipe.

of pressure p is replaced by the scaled *hydrostatic pressure* $-gz$, when positive z-axis is aligned vertically upwards. In that case G in (1.11) is replaced by $-g$, and flow moves downwards in the absence of pressure gradient.

Exercise: Determine the 2-D Pouisseuille flow; *i.e.* Steady unidirectional flow across two infinite parallel plates driven by a pressure gradient:

Exercise: Determine a steady uni-directional flow across two infinite parallel plates, where there is no pressure gradient but the upper-plate moves with velocity $U\hat{x}_1$. This is called the plane Coutte-flow.

2. An existence proof of N-S Initial value problem for $\Omega = \mathbb{R}^n$ in Fourier-Space

Recall the non-dimensional incompressible constant density Navier-Stokes equation

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \ , \text{with } u(x,0) = u_0(x) \tag{2.14}$$

$$\nabla \cdot u = 0 \tag{2.15}$$

Here we are dropping the vector notation. It is understood that $u(\cdot,t), f(\cdot,t) : \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$ and $p(\cdot,t) : \mathbb{R}^n \to \mathbb{R}$, and for a vector u, $|u|$ refers to its Euclidean norm, unless stated otherwise. If $\nabla \cdot f \neq 0$, we transform

$$f = \tilde{f} - \nabla \Phi \text{ where } -\Delta \Phi = \nabla \cdot f \tag{2.16}$$

From construction, $\nabla \cdot \tilde{f} = 0$. We note that the transformation (2.16) in (2.14) has the effect of replacing (f, p) by $(\tilde{f}, p + \Phi)$. Thus, without any loss of generality, we may assume $\nabla \cdot f = 0$.

Formally a Fourier Transform of (2.14) in $x \in \mathbb{R}^n$, with $\hat{u}(k,t) = \mathcal{F}\left[u(.,t)\right](k)$, using $\mathcal{F}\left[\partial_{x_j} v\right](k) = i k_j \mathcal{F}[v]$, $\mathcal{F}[gh](k) = \mathcal{F}[g] * \mathcal{F}[h]$, we obtain

$$\hat{u}_t - \nu |k|^2 \hat{u} = -i k_j \hat{u}_j * \hat{u} - i k \hat{p} + \hat{f}_k \ , \text{with } \hat{u}(k,0) = \hat{u}_0(k) \tag{2.17}$$

where $*$ denotes the Fourier Convolution, *i.e.*

$$\left[\hat{g} * \hat{h}\right](k) = \int_{k' \in \mathbb{R}^n} \hat{g}(k') \hat{h}(k - k') dk' \tag{2.18}$$

Note that $k \in \mathbb{R}^n$ and k_j is the j-th component of and repeated index refers to summation. Further (2.15) implies in Fourier space:

$$\hat{k} \cdot \hat{u} = 0 \tag{2.19}$$

Taking dot product of (2.17) with respect to $\hat{k}$, and using (2.18), we obtain

$$-k \cdot \hat{R} = |k|^2 \hat{p} \text{ , where } \hat{R} = k_j \hat{u}_j * \hat{u} \tag{2.20}$$

Therefore, with the above relation, we can eliminate $\hat{p}$ all together in (2.17) to obtain

$$\hat{u}_t - \nu |k|^2 \hat{u} = -i P_k \left[k_j \hat{u}_j * \hat{u} \right] + \hat{f} \tag{2.21}$$

where operator P_k is defined by

$$P_k \left[\hat{R} \right] = \left[I - \frac{k}{|k|^2} (k \cdot) \right] \hat{R} = \hat{R} - \frac{k}{|k|^2} \left(k \cdot \hat{R} \right), \tag{2.22}$$

Note from above the property that for any $\hat{h}$, $k \cdot P_k \left[\hat{h} \right] = 0$. P_k is the representation in Fouier-Space of the Hodge Projection $\mathcal{P}$ of a vector to the space of divergence-free vector fields (more on it later). From geometric considerations, it is apparent that $\left| (I - P_k) \hat{h}(k) \right|^2 + \left| P_k \hat{h}(k) \right|^2 = \left| \hat{h}(k) \right|^2$. Thus $P_k : L^2 \to L^2$ and $\| P_k \hat{R}_k \|_2 \leq \| \hat{R}_k \|_2$, where $\|.\|_2$ refers to the L^2 norm. We also note the same property in the L^1 norm or L^∞ norm in Fourier space. Inverting the differential operator on the left of (2.21) for given initial condition, we obtain an equivalent nonlinear integral equation for $\hat{u}$:

(2.23)

$$\hat{u}(k,t) = \hat{u}^{(0)}(k,t) + \int_0^t e^{-\nu^2 (t-\tau)} \left[-i k_j \hat{u}_j * \hat{u} \right] (k,\tau) d\tau =: \mathcal{N} \left[\hat{u} \right] (k,t),$$

where

$$\hat{u}^{(0)}(k,t) = e^{-\nu |k|^2 t} \hat{u}_0(k) + \int_0^t e^{-\nu |k|^2 (t-\tau)} \hat{f}(k,\tau) d\tau \tag{2.24}$$

Using contraction mapping theorem, we will prove that (2.23) has a unique solution locally in time in some ball in a suitable function space consistent with a continuous solution in time. On inverse Fourier Transform, this generates a smooth solution of Navier Stokes (2.14) locally in time.

Definition 2.1. *For $n = 2, 3$, $\beta \geq 0$ and $\mu > n$, we define norm $\|.\|_{\mu,\beta}$ so that*

$$\| \hat{f} \|_{\mu,\beta} = M \sup_{k \in \mathbb{R}^n} (1 + |k|)^\mu e^{\beta |k|} \left| \hat{f}(k) \right|, \tag{2.25}$$

where

$$M = \left\{ \sup_{k\in\mathbb{R}^n} (1+|k|)^\mu \int_{k'\in\mathbb{R}^n} \frac{1}{(1+|k-k'|)^\mu \, (1+|k'|)^\mu} dk' \right\}^{1/2}$$

For a function $\hat{v}$ of k and t, we define norm $\|.\|$ so that

$$\|\hat{v}\| = \sup_{t\in[0,T]} \|\hat{v}(\cdot,t)\|_{\mu,\beta} = \sup_{t\in[0,T],k\in\mathbb{R}^n} M \, (1+|k|)^\mu \, e^{\beta|k|} \, |\hat{u}(k,t)| \tag{2.26}$$

We define $\mathcal{S}$ to be Banch space of continuous functions in k and t such that $\|.\| < \infty$.

Lemma 2.2. *For any $\hat{f}$, $\hat{g}$,*

$$\|\hat{f} * \hat{g}\|_{\mu,\beta} \le \|\hat{f}\|_{\mu,\beta} \|\hat{g}\|_{\mu,\beta}$$

Proof. Using definition of $\|.\|_{\mu,\beta}$,

$$\Big| \int_{k'\in\mathbb{R}^n} \hat{f}(k')\hat{g}(k-k')dk' \Big| \le \frac{1}{M^2} \|\hat{f}\|_{\mu,\beta} \|\hat{g}\|_{\mu,\beta} \int_{k'\in\mathbb{R}^n} \frac{e^{-\beta(|k'|+|k-k'|)} dk'}{(1+|k'|)^\mu (1+|k-k'|)^\mu}$$

Since $|k-k'| + |k'| \le |k|$, $e^{-\beta(|k'|+|k-k'|)} \le e^{-\beta|k|}$. From this and the definition of $\|.\|_{\mu,\beta}$, Lemma follows provided $M < \infty$ exists. To show M exists, we may decompose

$$\int_{k'\in\mathbb{R}^n} \frac{dk'}{(1+|k'|)^\mu (1+|k-k'|)^\mu} = \left\{ \int_{|k'|<|k|/2} + \int_{|k'|\ge|k|/2} \right\} \frac{dk'}{(1+|k'|)^\mu (1+|k-k'|)^\mu}$$

$$\le \int_{|k'|\le|k|/2} \frac{1}{(1+|k|/2)^\mu} \frac{dk'}{(1+|k'|)^\mu} + \int_{|k-k'|\le|k|/2} \frac{1}{(1+|k|/2)^\mu} \frac{dk'}{(1+|k-k'|)^\mu}$$

$$\le \frac{2}{(1+|k|/2)^\mu} \int_{k'\in\mathbb{R}^n} \frac{dk'}{(1+|k'|)^\mu} \le \frac{C(\mu)}{(1+|k|)^\mu},$$

We note that using spherical (polar) coordinates, we can reduce $\int_{k'\in\mathbb{R}^n} \frac{dk'}{(1+|k'|)^\mu}$ to a one dimensional integral which exists for $\mu > n$ and this implies M is finite. ∎

Theorem 2.1. *(***Local Existence of NS***) If $\|\hat{u}_0\|_{\mu,\beta}, \sup_t \|\hat{f}(\cdot,t)\|_{\mu,\beta} < \infty$, then for sufficiently small T (taken ≤ 1), there exists unique solution of* (2.17) *for $\hat{u} \in \mathcal{S}$. For $\mu > n+2$, this corresponds to a classical solution of Navier Stokes (2.14)-(2.15).*

Proof. We note that

$$\|\hat{u}_0(k) e^{-|k|^2 t}\| \le \|\hat{u}_0\|_{\mu,\beta} \tag{2.27}$$

Further,

$$\Big\| \int_0^t e^{-|k|^2(t-\tau)} \hat{f}(k,\tau) \Big\|_{\mu,\beta} \le \sup_t \|\hat{f}(.,t)\|_{\mu,\beta} T \tag{2.28}$$

From (2.24)

$$(2.29)\quad \|\hat{u}^{(0)}\| \leq \|u_0\|_{\mu,\beta} + T \sup_{t\geq 0} \|\hat{f}(\cdot,t)\|_{\mu,\beta} \leq \|\hat{u}_0\|_{\mu,\beta} + \sup_{t\geq 0} \|\hat{f}(\cdot,t)\|_{\mu,\beta}$$

Further, from Lemma 2.2 and definition of $\|.\|$ in (2.26),

$$(2.30)\quad \Big| \int_0^t e^{-\nu|k|^2(t-\tau)} ik_j \left[\hat{u}_j * \hat{u}\right](k,\tau) d\tau \Big|$$

$$\leq \|u\|^2 e^{-\beta|k|} \int_0^t e^{-\nu|k|^2(t-\tau)} \frac{|k|}{M(1+|k|)^\mu} d\tau \leq \|\hat{u}\|^2 \frac{e^{-\beta|k|}}{M\nu|k|(1+|k|)^\mu} \left[1 - e^{-\nu|k|^2 t}\right]$$

Therefore, it follows that

(2.31)

$$\| \int_0^t e^{-\nu|k|^2(t-\tau)} ik_j \left[\hat{u}_j * \hat{u}\right](k,\tau) d\tau \| \leq \|\hat{u}\|^2 \sqrt{\frac{T}{\nu}} \left(\sup_{\gamma>0} \frac{1-e^{-\gamma}}{\sqrt{\gamma}} \right) \equiv c\sqrt{\frac{T}{\nu}} \|\hat{u}\|^2$$

$$(2.32)\qquad \|\mathcal{N}[\hat{u}]\| \leq \|\hat{u}^{(0)}\| + c\sqrt{\frac{T}{\nu}} \|u\|^2,$$

In a similar manner,

$$(2.33)\qquad \|\mathcal{N}[\hat{u}_1] - \mathcal{N}[\hat{u}_2]\| \leq c\sqrt{\frac{T}{\nu}} \left(\|\hat{u}_1\| + \|\hat{u}_2\|\right) \|\hat{u}_1 - \hat{u}_2\|,$$

From the above equations, that if

$$(2.34)\qquad T < \frac{\nu}{16c^2 \|\hat{u}^{(0)}\|^2} = \frac{\nu}{16c^2 \left[\|\hat{u}_0\|_{\mu,\beta} + \sup_{t\geq 0} \|\hat{f}\|_{\mu,\beta}\right]^2},$$

then $\mathcal{N} : B \to B$ contractively, where $B \in \mathcal{S}$ is a ball of size $2\|u^{(0)}\|$. Therefore, from contraction mapping theorem in a Banach space, there exists unique solution to the integral equation (2.23) and hence (2.17), T small enough to satisfy (2.34). On Fourier transforming (2.17), which exists for $\mu > n+2$, we obtain a classical smooth solution of Navier-Stokes (2.14)-(2.15). ∎

Remark 2.3. The existence time T depends on initial condition u_0 and forcing f in a bad way. Larger data gives smaller existence time. This prevents continuation of the local existence argument to get global existence results. Also, there is dependence on inverse Reynolds number, *i.e.* nondimensional viscosity ν in the proof. This is to be expected since the proof relies essentially on inversion of heat operator involving viscosity. For larger time the advection term $\mathbf{u} \cdot \nabla$ becomes more important. Later in the course, we will note energy methods which works both for N-S and Euler equations. Indeed, with help from potential theory, these methods help establish global existence in 2-D. Also note

$\beta > 0$ implies that solution $u(x,t)$ in the physical domain is a real analytic function of x, with analyticity width β remaining the same through out the existence time.

3. Global existence for small data or large ν for $x \in \mathbb{T}^n$, for $n = 2, 3$

We now consider Navier-Stokes equation in a $[0, 2\pi]^n$ periodic box, *i.e.* for $n = 3$,

(3.35)
$$u(x_1+2\pi, x_2, x_3) = u(x_1, x_2+2\pi, x_3) = u(x_1, x_2, x_3+2\pi) = u(x_1, x_2, x_3)$$

with forcing $f = 0$ and initial condition u_0 periodic and divergence free. Then as before for $x \in \mathbb{R}^n$, we obtain in the Fourier Space the same equations (2.21) and (2.23), with $\hat{f} = 0$, except now $k \in \mathbb{Z}^n$ and Fourier convolution $*$ now involves sum of $k' \in \mathbb{Z}^n$. From (2.21), for $k = 0$, we note

$$\partial_t \hat{u} = 0 \ , \ \text{implying } \hat{u}(0,t) = \hat{u}_0(0) \tag{3.36}$$

Therefore, by translating in a right frame of reference, we may choose, without any loss of generality,

$$\hat{u}(0,t) = 0, \tag{3.37}$$

We define norm $\|.\|_{\mu,\beta}$ as before for $\mu > n$ and $\beta \geq 0$ in (2.25), except that $k \in \mathbb{Z}^n$ and

$$M^2 = \sup_{k \in \mathbb{Z}^n} (1+|k|)^\mu \sum_{k' \in \mathbb{Z}^n} \frac{1}{(1+|k'|)^\mu (1+|k-k'|)^\mu} \tag{3.38}$$

Lemma 3.1. *For the discrete convolution $*$ operation*

$$\|\hat{f} * \hat{g}\|_{\mu,\beta} \leq \|\hat{f}\|_{\mu,\beta} \|\hat{g}\|_{\mu,\beta}$$

Proof. From definition of discrete convolution

(3.39)
$$\Big| \sum_{k' in \mathbb{Z}^n} \hat{f}(k') \hat{g}(k-k') \Big| \leq \frac{1}{M^2} \|\hat{f}\|_{\mu,\beta} \|\hat{g}\|_{\mu,\beta} \sum_{k' \in \mathbb{Z}^n} \frac{e^{-\beta|k'| - \beta|k-k'|}}{(1+|k'|)^\mu (1+|k-k'|)^\mu}$$

Since $e^{-\beta(|k'|+|k-k'|)} \leq e^{-\beta|k|}$ as before for the continuous case, the Lemma follows from definition of M in (3.38), provided $M^2 < \infty$. To show this, as before, with the integral, we break up k' into two sets: $S_1 = \{k' : |k'| > |k|/2\}$ and $S_2 = \{k' : |k'| \leq |k|/2\}$. Then

(3.40)
$$\sum_{k' \in S_1} \frac{1}{(1+|k'|)^\mu (1+|k-k'|)^\mu} \leq \frac{1}{(1+|k|/2)^\mu} \sum_{k' \in \mathbb{Z}^n} \frac{1}{(1+|k-k'|)^\mu} = \frac{1}{(1+|k|/2)^\mu} \sum_{\tilde{k} \in \mathbb{Z}^n} \frac{1}{(1+|\tilde{k}|)^\mu}$$

(3.41)
$$\sum_{k'\in S_2}\frac{1}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\leq\sum_{k-k'\in S_1}\frac{1}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\leq\frac{1}{(1+|k|/2)^\mu}\sum_{k'\in\mathbb{Z}^n}\frac{1}{(1+|k'|)^\mu}$$

where $\sum_{k'\in\mathbb{Z}^n}(1+|k'|)^{-\mu}$ converges from integral test. Therefore,

$$(1+|k|)^\mu\left(\sum_{k'\in S_1}\frac{1}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\right)\leq M^2<\infty \tag{3.42}$$

for some M depending on μ but independent of k. We define M to be the smallest such M. ∎

Definition 3.2. *We now introduce a weighted space time norm*

$$\|\hat{u}\|_E=\sup_{t\in[0,T]}e^{\nu t}\|\hat{u}(.,t)\|_{\mu,\beta} \tag{3.43}$$

We denote by $\mathcal{S}_E$ to be the Banach space of functions with $\|.\|_E<\infty$.

Theorem 3.1. Global existence for unforced case *For $n=2,3$, if for $\mu>n$, $\beta\geq 0$, $\|\hat{u}_0\|_{\mu,\beta}<\frac{\nu}{4\sqrt{2}}$, then there exists globally unique solution to NS in $\mathcal{S}_E$.*

Proof. We have in this case $\hat{u}^0(k,t)=\hat{u}_0e^{-\nu|k|^2t}$ and therefore, since $|k|\geq 0$ for all nonzero $\hat{u}$, we have

$$\|\hat{u}^{(0)}\|_E\leq\|\hat{u}_0\|_{\mu,\beta} \tag{3.44}$$

Further, from definition of $\|.\|_E$ and Lemma 3.1,

(3.45)
$$\left|-ik_j\int_0^t e^{-|k|^2(t-\tau)}\left[\hat{u}_j*\hat{u}\right](k,\tau)d\tau\right|\leq\frac{e^{-\nu t}e^{-\beta|k|}\|\hat{u}\|_E^2}{M(1+|k|)^\mu}\int_0^t|k|e^{-\nu(|k|^2-1)(t-\tau)}e^{-\nu\tau}d\tau,$$

There are two cases, $|k|=1$ and $|k|>1$. For $|k|=1$, we note that

$$\int_0^t|k|e^{-\nu(|k|^2-1)(t-\tau)}e^{-\nu\tau}d\tau\leq\frac{1}{\nu} \tag{3.46}$$

We note $\inf_{k\in\mathbb{Z}^n,|k|>1}|k|=\sqrt{2}$, and so for $|k|>1$,

(3.47)
$$\int_0^t|k|e^{-\nu(|k|^2-1)(t-\tau)}e^{-\nu\tau}d\tau\leq\int_0^t|k|e^{-\nu(|k|^2-1)(t-\tau)}d\tau\leq\frac{|k|}{\nu(|k|^2-1)}\leq\frac{\sqrt{2}}{\nu}$$

we obtain

$$\|-ik_j\int_0^t e^{-|k|^2(t-\tau)}\left[\hat{u}_j*\hat{u}\right](k,\tau)d\tau\|_E\leq\frac{\sqrt{2}}{\nu}\|\hat{u}\|_E^2, \tag{3.48}$$

Therefore, with $\mathcal{N}$ defined in (2.23), we obtain for $\hat{u},\hat{u}_1,\hat{u}_2\in\mathcal{S}_E$,

$$\|\mathcal{N}\left[\hat{u}\right]\|\leq\|\hat{u}_0\|_{\mu,\beta}+\frac{\sqrt{2}}{\nu}\|\hat{u}\|_E^2, \tag{3.49}$$

and in a similar manner,

$$\|\mathcal{N}[\hat{u}_1] - \mathcal{N}[\hat{u}_2]\| \leq \frac{\sqrt{2}}{\nu}\left(\|\hat{u}_1\| + \|\hat{u}_2\|_E\right)\|\hat{u}_1 - \hat{u}_2\|_E \tag{3.50}$$

From above, it is easily seen that $\mathcal{N} : B_E \to B_E$ in a ball $B_E \subset \mathcal{S}_E$ of size $2\|\hat{u}_0\|_{\mu,\beta}$ if

$$\frac{4\sqrt{2}}{\nu}\|\hat{u}_0\|_{\mu,\beta} < 1 \tag{3.51}$$

From contraction mapping theorem, if initial data is small enough or ν large enough to satisfy (3.51), then there exists unique solution solution $\hat{u} \in B_E$. Since this argument is valid for any T, this is the solution is global in time. ∎

Remark 3.3. *Note that though we argued that there is a unique solution only in a Ball B_E, this can be the only solution in $\mathcal{S}_E$. This is because from continuity in time, solution must be in a ball of size $2\|\hat{u}_0\|_{\mu,\beta}$, which it never escapes. Also, though the solution is shown only to be in the space $\mathcal{S}_E$, it turns out that for $t > 0$, there is instantaneous smoothing in the sense that for $t > 0$, $\|u(.,t)\|_{\mu+2,\beta} < \infty$. We will show this property next time.*