Week 5 Notes, Math 865, Tanveer

1. Instantaneous smoothing of unforced NS solutions for $N = 2, 3$

Recall last week we found solutions to

(1.1)
$$\hat{u}(k,t) = \hat{u}_0(k) e^{-\nu|k|^2 t} + \int_0^t -ik_j P_k \left[\hat{u}_j * \hat{u}\right](k,\tau) e^{-\nu|k|^2(t-\tau)} d\tau =: \mathcal{N}\left[\hat{u}(k,t)\right]$$

either for $k \in \mathbb{R}^N$ or $k \in \mathbb{Z}^N$ for which $\|\hat{u}(\cdot,t)\|_{\mu,\beta}$ is finite for $\mu > n$, $\beta \geq 0$. We now prove that the solution is instantaneously smoothed, *i.e.* for $t > 0$, $\|\hat{u}(\cdot,t)\|_{\mu+2,\beta} < \infty$, implying that $u(x,t) = \mathcal{F}^{-1}\left[\hat{u}(\cdot,t)\right](x)$ is a classical solution of NS equations.

Definition 1.1. *For $T > \epsilon > 0$, define $W_\epsilon(k) = \sup_{t \in [\epsilon, T]} \left|\hat{u}(k,t)\right|$, where $\hat{u}(.,t)$ is a solution of (1.1) for $t \in [0,T]$.*

Lemma 1.2. *The solution $\hat{u}(\cdot,t)$ satisfying (1.1) satisfies $\|\hat{u}(;t)\|_{\mu+2,\beta} < \infty$ for any $t \in (0,T]$.*

Proof. From (1.1), it immediately follows that

(1.2)
$$|k||\hat{u}(k,t)| \leq |k|\hat{u}_0(k) e^{-\nu|k|^2 t} + \frac{e^{-\beta|k|}}{M(1+|k|)^\mu} \|\hat{u}\|^2 \int_0^t |k|^2 e^{-\nu|k|^2(t-\tau)} d\tau$$

It follows that

$$\|kW_\epsilon(k)\|_{\mu,\beta} \leq \left(\sup_{\gamma>0} \gamma e^{-\gamma^2}\right) \nu^{-1/2} \epsilon^{-1/2} \|\hat{u}_0\|_{\mu,\beta} + \frac{1}{\nu}\|\hat{u}\|^2, \tag{1.3}$$

implying that $\||k|\hat{u}(.,t)\|_{\mu,\beta} < \infty$ for $t \in [\epsilon, T]$. We note that using the NS equation is autonomous in time and starting the clock at $t = \epsilon$, and using Fourier transform of $u_j \partial_{x_j} u$ instead of $\partial_{x_j}\left[u_j u\right]$, we may rewrite the integral form of NS equation in the following form for $t \in [\epsilon, T]$. in the form

(1.4)
$$|k|^2 \hat{u}(k,t) = |k|^2 \hat{u}(k,\epsilon) e^{-\nu|k|^2(t-\epsilon)} + \int_\epsilon^t e^{-\nu|k|^2(t-\tau)} |k|^2 P_k \left[\hat{u}_j * (k_j \hat{u})\right](k,\tau) d\tau$$

Repeating the same argument as above for $t \in [2\epsilon, T]$, we have

(1.5)
$$\||k|^2 W_{2\epsilon}(k)\|_{\mu,\beta} \leq \frac{1}{\nu\epsilon}\left(\sup_{\gamma>0} \gamma^2 e^{-\gamma}\right) \|\hat{u}(.,\epsilon)\|_{\mu,\beta} + \frac{1}{\nu}\|kW_\epsilon\|_{\mu,\beta} \|W_\epsilon(k)\|_{\mu,\beta} < \infty,$$

implying $\||k|^2 \hat{u}(.,t)\|_{\mu,\beta} < \infty$ for $t \in [2\epsilon, T]$. Since $\epsilon > 0$ is arbitrary, the Lemma follows. ∎

2. Energy Methods for Euler and Navier-Stokes Equation

We will consider this week basic energy estimates. These are estimates on the L_2 spatial norms of the solution $u(x, t)$ and its higher deriviatives with respect to x. Like other PDE initial value problems, these estimates are most useful in establishing existence and uniqueness of solutions.

For simplicity, we will first take $\Omega = \mathbb{R}^N$, where $N = 2$ or 3. Later in class, we will consider the case with boundaries. The exposition of this topic is close to Bertozzi & Majda (See Reference), though with some differences in notation.

2.1. **Basic Definitions.**

Definition 2.1. *For $v \in \mathbb{R}^N$,*

$$|v| = \left(\sum_{j=1}^{N} v_j^2 \right)^{1/2} \tag{2.6}$$

For a function $f : \mathbb{R}^N \to \mathbb{R}^N$,

$$Df = \left(\partial_{x_1} f, .., \partial_{x_N} f \right), \tag{2.7}$$

with each component $\partial_{x_j} f \in \mathbb{R}^N$.

$$|Df| = \left(\sum_{i,j=1}^{N} [\partial_{x_j} f_i]^2 \right)^{1/2} \tag{2.8}$$

Analogously, higher order tensors $D^2 f$, $D^3 f$ and their absolute values are defined. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..\alpha_N)$, each being non-negative integers, we define

$$D^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} .. \partial_{x_N}^{\alpha_N} f \tag{2.9}$$

We define the norm of the multi-index α:

$$|\alpha| = \alpha_1 + \alpha_2 + .. + \alpha_N \tag{2.10}$$

We also consider norms

$$\|f\|_0 \equiv \|f\|_{L_2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |f(x)|^2 dx \right)^{1/2} \tag{2.11}$$

The corresponding L_2 inner product will be denoted by

$$(f, g)_0 = \int_{\mathbb{R}^N} f(x) g(x) dx \tag{2.12}$$

We define higher order energy norms $\|.\|_m{}^{(1)}$*:*

$$\|f\|_m \equiv \|f\|_{H_m(\mathbb{R}^N)} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|_0^2 \right)^{1/2} \tag{2.13}$$

The corresponding inner-product in H_m *will be denoted by* $(.,.)_m$

2.2. **Calculus Inequalties for Sobolev Spaces and Mollifiers.** We have already introduced the Sobolev space $H_m(\mathbb{R}^N)$ for integer $m \ge 0$. We now extend it to $H_s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$. In the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of smooth functions with rapid decay at ∞, we introduce the norm

$$\|u\|_s = \left\{ \int_{\mathbb{R}^N} (1 + |k|)^{2s} \hat{u}(k) dk \right\}^{1/2} \tag{2.14}$$

where $\hat{u}(k) = \mathcal{F}[u](k)$, *i.e.* the Fourier-Transform of u. The completion of $\mathcal{S}(\mathbb{R}^N)$ with norem (2.14) will be referred to as $H_s(\mathbb{R}^N)$. You can check that for $s = m$, that this is equivalent to the original definition of H_m.

One of the most important Sobolev space property that we will use is the Sobolev inequality below:

Lemma 2.2. *Sobolev embedding Theorem*

The space $H_{s+k}(\mathbb{R}^n)$*, for* $s > N/2$*,* $k \in \mathbb{Z}^+ \cup \{0\}$ *is continuously embedded in the space* $\mathbf{C}^k(\mathbb{R}^N)$*, and there exists a constant* $c > 0$ *such that*

$$|v|_{\mathbf{C}^k} \le c \|v\|_{s+k} \ , \text{ for any } \ v \in H_{s+k}(\mathbb{R}^N) \tag{2.15}$$

Some other calculus inequalities in the following Lemma will be useful for our purposes:

Lemma 2.3. i. *For all* $m \in \mathbb{Z}^+ \cup \{0\}$*, there exists* $c > 0$ *such that for all* $u, v \in L_\infty \cap H_m(\mathbb{R}^N)$*,*

$$\|uv\|_m \le c \left\{ \|u\|_\infty \|D^m v\|_0 + \|D^m u\|_0 \|v\|_\infty \right\}$$

$$\sum_{0 \le |\alpha| \le m} \|D^\alpha (uv) - u D^\alpha v\|_0 \le c \left\{ \|\nabla u\|_\infty \|D^{m-1} v\|_0 + \|D^m u\|_0 \|v\|_\infty \right\}$$

ii. *For all* $s > N/2$*,* $H_s(\mathbb{R}^N)$ *is a Banach algebra, i.e. there exits a constant* c *so that for all* $u, v \in H_s(\mathbb{R}^N)$*,*

$$\|uv\|_s \le c \|u\|_s \|v\|_s$$

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(1) Note that through a Fourier-representation, $\|.\|_m$ can be generalized to nonintegral or negative m. We will use such generalizations later.

We now introduce a mollifier. Suppose $\rho = \rho(|x|) \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, *i.e.* infinitely smooth function with compact support. Suppose also $\rho \geq 0$, and $\int_{\mathbb{R}^N} \rho dx = 1$. Then we define mollification of v, denoted by $\mathcal{I}_\epsilon v$ to be a function given by:

$$(\mathcal{I}_\epsilon v)(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) v(y) dy \tag{2.16}$$

Lemma 2.4. *Properties of Mollifier*

Let $\mathcal{I}_\epsilon$ be the mollifier defined in (2.16). Then $\mathcal{I}_\epsilon v \in \mathcal{C}^\infty(\mathbb{R}^N)$ and

i. *for all $v \in \mathcal{C}^0(\mathbb{R}^N)$, $\mathcal{I}_\epsilon v \to v$ uniformly on any compact set $\Omega \subset \mathbb{R}^N$ and*

$$\|\mathcal{I}_\epsilon v\|_\infty \leq \|v\|_\infty$$

ii. *Mollifiers commute with distribution derivatives*

$$D^\alpha \mathcal{I}_\epsilon v = \mathcal{I}_\epsilon D^\alpha v \text{ for any } |\alpha| \leq m, \ v \in H_m$$

iii. *For all $u \in L_p(\mathbb{R}^N)$, $v \in L_q(\mathbb{R}^N)$, $1/p + 1/q = 1$,*

$$\int_{\mathbb{R}^N} (\mathcal{I}_\epsilon u) v dx = \int_{\mathbb{R}^N} u (\mathcal{I}_\epsilon v) dx$$

iv. *For all $v \in H_s(\mathbb{R}^N)$, $\mathcal{I}_\epsilon v$ converges to v in H_s and the rate of convergence in the H_{s-1} norm is linear in ϵ, i.e.*

$$\lim_{\epsilon \to 0^+} \|\mathcal{I}_\epsilon v - v\|_s = 0$$

$$\|\mathcal{I}_\epsilon v - v\|_{s-1} \leq C\epsilon \|v\|_s$$

v. *For all $v \in H_m(\mathbb{R}^N)$, $k \in \mathbb{Z}^+ \cup \{0\}$, and $\epsilon > 0$,*

$$\|\mathcal{I}_\epsilon v\|_{m+k} \leq \frac{c_{mk}}{\epsilon^k} \|v\|_m$$

$$\|\mathcal{I}_\epsilon D^k v\|_\infty \leq \frac{c_k}{\epsilon^{N/2+k}} \|v\|_0$$

3. Hodge Projection and Properties

Lemma 3.1. *Any vector field $v \in H_m(\mathbb{R}^N)$ for $m \in \mathbb{Z}^+ \cup \{0\}$ has a unique orthogonal decomposition*

$$v = \nabla\phi + w \ , \text{ where } \nabla\phi \ , w \in H_m \ , \nabla \cdot w = 0$$

We define $w = \mathcal{P}v$ as the Hodge projection of v onto the divergence free vector field. Further,

i. $(\mathcal{P}v, \nabla\phi)_m = 0$ *and* $\|\mathcal{P}v\|_m^2 + \|\nabla\phi\|_m^2 = \|v\|_m^2$.

ii. $\mathcal{P}$ *commutes with D^α in H_m for $|\alpha| \leq m$: $\mathcal{P}D^\alpha v = D^\alpha \mathcal{P} v$.*

iii. $\mathcal{P}\mathcal{I}_\epsilon v = \mathcal{I}_\epsilon P v$

iv. $\mathcal{P}$ *is symmetric:* $(\mathcal{P}u, v)_m = (u, \mathcal{P}v)_m$

Proof. We only consider $m = 0$. Other cases follow simply by noting property **ii**: that $\mathcal{P}$ commutes with D. We further consider only $v \in \mathbf{C}_c^\infty(\mathbb{R}^N)$. This space is dense in H_m and hence all the results will follow for more general v, except that derivatives have to be understood in the sense of a distribution. Define ϕ by solving

$$\Delta\phi = \nabla \cdot v \ , \ \text{with } \phi \to 0 \text{ as } x \to \infty$$

Using Green's function for Laplacian, we know

$$\phi(x) = \left[\Delta^{-1}\nabla \cdot v\right](x) = \int_{y \in \mathbb{R}^N} G(x-y)(\nabla \cdot v)(y) dy, \tag{3.17}$$

where

$$G(x) = \frac{1}{2\pi} \log|x| \text{ for } N = 2, \ G(x) = -\frac{1}{4\pi|x|} \text{ for } N = 3$$

Then it is clear that

$$\nabla\phi(x) = \int_{y \in \mathbb{R}^N} \nabla G(x-y)(\nabla \cdot v)(y) \equiv \left[\nabla\Delta^{-1}\nabla \cdot v\right](x) \tag{3.18}$$

Now, notice that as $x \to \infty$,

$$\nabla\phi(x) \sim [\nabla G](x) \int_{y \in \mathbb{R}^N} (\nabla \cdot v)(y) dy + O(|x|^{-N})$$

From applying Gauss's theorem on the first term,

$$\nabla\phi(x) = O(|x|^{-N}) \text{ as } |x| \to \infty,$$

and hence $\nabla\phi \in L_2(\mathbb{R}^N)$. Define

$$\mathcal{P}v = w = v - \nabla\phi$$

Clearly since $v, \nabla\phi \in L_2(\mathbb{R}^N)$, so is $w = \mathcal{P}v$. It is clear that

$$\nabla \cdot w = \nabla \cdot v - \Delta\phi = 0$$

So, $\mathcal{P}v$ is divergence free, and from the decay rate of $\nabla\phi$ for large x, it follows that

$$w \sim O(|x|^{-N}) \text{ as } |x| \to \infty$$

Now, property **i.** follows since

$$(w, \nabla\phi)_0 = \int_{\mathbb{R}^N} w_j \partial_{x_j} \phi dx = \int_{\mathbb{R}^N} \partial_{x_j}(w_j \phi) = \lim_{R \to \infty} \int_{|x|=R} \phi(w \cdot n) dx = 0$$

since for large x, $\phi = O(\log|x|)$ for $N = 2$ and $\phi = O(|x|^{-N+2})$ for $N = 3$, while $w = O(|x|^{-N})$. Also,

$$\|v\|_0^2 = (w + \nabla\phi, w + \nabla\phi)_0 = (w, w) + (\nabla\phi, \nabla\phi)_0 = \|\mathcal{P}v\|_0^2 + \|\nabla\phi\|_0^2$$

because of the orthogonality property.

Property **ii.** follows simply from the observation that

$$D^\alpha \mathcal{P} v = D^\alpha w = D^\alpha v - D^\alpha \nabla \phi = D^\alpha v - \nabla D^\alpha \phi = \mathcal{P} D^\alpha v,$$

since

$$\Delta (D^\alpha \phi) = \nabla \cdot (D^\alpha v)$$

Property **iii.** follows from the commuting property of $\mathcal{I}_\epsilon$ with Δ^{-1} (defined in (3.17)) and with any differential operator, since

$$\mathcal{I}_\epsilon \mathcal{P} v = \mathcal{I}_\epsilon v - \mathcal{I}_\epsilon \nabla \Delta^{-1} \nabla \cdot v = (\mathcal{I}_\epsilon v) - \nabla \Delta^{-1} \nabla \cdot (\mathcal{I}_\epsilon v) = \mathcal{P} \mathcal{I}_\epsilon v$$

Property **iv** follows for $m = 0$ because

$$(\mathcal{P} v, u)_0 = \left(u, v - \nabla \Delta^{-1} \nabla \cdot v\right)_0 = (u, v)_0 + \left(\nabla u, \Delta^{-1} \nabla \cdot v\right)_0 = (u, v)_0 + \left(\Delta^{-1} \nabla u, \nabla \cdot v\right)_0$$
$$= (u, v)_0 - \left(\nabla \cdot (\Delta^{-1} \nabla u), v\right)_0 = \left(u - \nabla \Delta^{-1} \nabla \cdot u, v\right)_0 = (v, \mathcal{P} u)_0$$

▮

4. Energy dissipation and uniqueness arguments

Consider the incompressible constant density Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \tag{4.19}$$

$$\nabla \cdot u = 0, \tag{4.20}$$

where $\nabla \cdot b = 0$, without any loss of generality. Applying Hodge projection operator $\mathcal{P}$ on (4.19) to obtain

$$\partial_t u + \mathcal{P} \left[u \cdot \nabla u \right] = \nu \Delta u + b \tag{4.21}$$

We take inner product of (4.21) with u in $(.,)_0$ space and use $(u, u_t) = \frac{d}{dt} \frac{1}{2} \| u(., t) \|_0^2$,

$$(u, \mathcal{P} \left[u \cdot \nabla u \right])_0 = \left(u, u_j \partial_{x_j}\right) = \int_{x \in \mathbb{R}^N} \partial_{x_j} \left(u_j |u|^2 / 2\right) dx = 0, \tag{4.22}$$

for sufficiently rapidly decaying u (for *e.g.* $u = o(|x|^{-(N-1)/3})$ as $|x| \to \infty$ or even $u \in L_3(\mathbb{R}^N)$ will suffice. Note that from a Sobolev embedding result $H_{1/2}(\mathbb{R}^3) \subset L_3(\mathbb{R}^3)$).

$$\frac{d}{dt} \frac{1}{2} \| u(., t) \|_0^2 = -\nu \| \nabla u(., t) \|_0^2 + (u, b)_0 \tag{4.23}$$

The quantity $E(t) = \frac{1}{2} \| u(., t) \|_0^2$ is the *Kinetic energy* of the fluid at time t, while $\epsilon = \nu \| \nabla u(., t) \|_0^2$ is the *viscous dissipation rate of energy*, while $(u, b)_0$ is the *rate of work done* by force b. Thus (4.23) is simply a physical statement that the rate at which Kinetic energy changes is due to loss of energy due to dissipation and the gain of energy due to work done by force b. If the same argument is done with flow past a finite solid body, there will be additional contribution due to work done by the body on the fluid, which can be calculated as an exercise. The viscously dissipated energy is actually converted to heat. For *compressible* fluid flow, we have to couple heat energy and thermodynamics with momentum equation to get a complete set of equations. However, this is not necessary for *incompressible* flow.

Time integration of (4.23) leads to

$$\frac{1}{2}\|u(.,t)\|_0^2 + \nu \int_0^t \|Du(.,\tau)\|_0^2 d\tau = \int_0^t (b,u)_0(\tau) d\tau \tag{4.24}$$

Exercise: Calculate the Energy balance equation equivalent to (4.24) for a domain Ω exterior to to a stationary solid and identify the term which is the rate of work done by the body on the fluid. It might be better to use inner product of u with (4.19), since we are yet to discus Hodge projection $\mathcal{P}$ for finite domain.

4.1. **Energy Estimate, Uniqueness and ν dependence of Smooth Solutions.** Let u, w be two Navier-Stokes solution, corresponding to forcing b and c respectively. We assume b and c to be smooth as well and decaying sufficiently fast in x at ∞. We denote the corresponding pressures by p and q. Then consider the difference $v = u - w$. It is easy to check that v satisfies:

$$\partial_t v + v\cdot\nabla v + w\cdot\nabla v + v\cdot\nabla w = -\nabla P + \nu\Delta v + f \text{ , where } f = b - c, \ P = p - q \tag{4.25}$$

The i-th component of the above equation may be written as

$$\partial_t v_i + v_j \partial_{x_j} v_i + w_j \partial_{x_j} v_i + v_j \partial_{x_j} w_i = -\partial_{x_i} P + \nu \partial_{x_j}^2 v_i + f_i \tag{4.26}$$

Multiplying above by v_i and integrating we obtain that

$$\partial_t \frac{1}{2} v_i^2 + \frac{1}{2}\partial_{x_j}\left[(v_j + w_j) v_i^2\right] + v_i v_j \partial_{x_j} w_i = -\partial_{x_i}(u_i P) + \nu \partial_{x_j}\left(v_i \partial_{x_j} v_i\right) - \nu(\partial_{x_j} v_i)(\partial_{x_j} v_i) + v_i f_i \tag{4.27}$$

So integrating over $\mathbb{R}^N$ with usual assumptions on decay of velocity and pressure fields at ∞, we obtain by using

$$|(v,f)_0| \le \|v\|_0 \|f\|_0 \text{ , and } |(v, v\cdot\nabla w)_0| \le \|Dw(.,t)\|_\infty \|v\|_0^2 \text{ ,} \tag{4.28}$$

$$\frac{d}{dt}\frac{1}{2}\|v\|_0^2 + \nu\|Dv\|_0^2 \le \|Dw\|_\infty \|v\|_0^2 + \|v\|_0 \|f\|_0 \tag{4.29}$$

So, in particular,

$$\frac{d}{dt}\|v\|_0 \le \|\nabla w\|_\infty \|v\|_0 + \|f\|_0 \tag{4.30}$$

Also, on integrating (4.29) between $t = 0$ to $t = T$, we obtain

(4.31)
$$\frac{1}{2}\|v(.,T)\|_0^2+\nu\int_0^T \|Dv(.,t)\|_0^2 dt \le \frac{1}{2}\|v(.,0)\|_0^2+\int_0^T \|Dw(.,t)\|_\infty \|v(.,t)\|_0^2 dt+\int_0^T \|v(.,t)\|_0 \|f(.,t)$$

Using well-known Gronwall's inequality on (4.30) and the definition of v, we obtain the following Lemma

Lemma 4.1. *Let u and w be two smooth $L_2(\mathbb{R}^N)$ solutions to the Navier-Stokes equation for $t \in [0,T]$ for the same viscosity ν, but different forcing b and c respectively. Then,*

(4.32)
$$\sup_{t\in[0,T]} \|u(.,t)-w(.,t)\|_0 \le \left\{ \|u(.,0) - w(.,0)\|_0 + \int_0^T \|b(.,t) - c(.,t)\|_0 dt \right\} \exp\left[\int_0^T |\nabla w(.,t)\|_\infty d\right.$$

Corollary 4.2. *Uniqueness of smooth solutions*

Let $u(.,t)$ and $w(.,t)$ be two smooth $L_2(\mathbb{R}^N)$ solutions to incompressible constant density Navier-Stokes equation for $t \in [0,T]$ with same initial data and forcing. Then, the solution is unique.

Proof. This simply follows from Lemma 2.11, since $u(.,0) - w(.,0) = 0$ and $b - c = 0$. ∎

Remark 4.3. *The energy estimate (4.32) does not explicitly depend on ν and is equally valid for $\nu = 0$, i.e. for the Euler equation.*

The energy estimate (4.32) is also useful in estimating the difference between smooth Euler and Navier-Stokes solution with the same initial data and forcing. Let $u^{[0]}$ be a smooth solution to the Euler equation, *i.e.* $\nu = 0$, while $u^{[\nu]}$ is a solution to Navier-Stokes equation with the same initial data and forcing. Then, we can obtain an equation for $v = u^{[\nu]} - u^{[0]}$:

$$\partial_t v + v \cdot \nabla v + u^{[0]} \cdot \nabla v + v \cdot \nabla u^{[0]} = -\nabla P + \nu \Delta v + f \tag{4.33}$$
$$\text{where } f = \nu \Delta u^{[0]},\ P = p - q \text{ is the difference of pressure}$$

This is the same equation as for (4.25), with w replaced by $u^{[0]}$, and a different meaning of f. Therefore, the energy estimate (4.32) in this case becomes

(4.34)
$$\sup_{t\in[0,T]} \|u^{[\nu]}(.,t)-u^{[0]}(.,t)\|_0 \le \nu \left(\int_0^T \|\Delta u^{[0]}(.t)\|_0 dt\right) \exp\left\{\int_0^T \|Du^{[0]}(.,t)\|_\infty dt\right\} \le \nu T C(u^{[0]},T)$$

Notice that (4.31) with w replaced by $u^{[0]}$, and with $f = -\nu \Delta u^{[0]}$ gives rise to
(4.35)
$$\nu \int_0^T \|Dv(.,t)\|_0^2 dt \leq \int_0^T \|Du^{[0]}(.,t)\|_\infty \|v(.,t)\|_0^2 dt + \nu \int_0^T \|v(.,t)\|_0 \|\Delta u^{[0]}(.,t)\|_0 dt$$
Using estimate (4.34) estimate, we obtain,
(4.36)
$$\int_0^T \|Dv(.,t)\|_0^2 dt \leq \nu T C \left\{ CT \int_0^T \|Du^{[0]}(.,t)\|_\infty dt + \nu \int_0^T \|\Delta u^{[0]}(.,t)\|_0 dt \right\} \leq \nu T^2 c_2(u^{[0]}, T)$$
So,
(4.37)
$$\int_0^T \|Dv(.,t)\|_0 dt \leq T^{1/2} \left(\int_0^T \|Dv(.,t)\|^2 dt \right)^{1/2} \leq \nu^{1/2} T^{3/2} C_2(u^{[0]}, T)$$
This implies the following proposition:

Proposition 4.4. *Comparison of smooth Euler and Navier-Stokes Solution*

Given the same initial data and forcing, then the difference v between smooth $L_2(\mathbb{R}^N)$ Navier-Stokes and Euler solution over a common interval of existence $[0, T]$ satisfies (4.34) and (4.37). In particular for any fixed T, as $\nu \to 0$, $u^{[\nu]}(.,t) \to u^{[0]}(.,t)$, and $Du^{[\nu]}(.,t) \to Du^{[0]}(.,t)$ uniformly for $t \in [0, T]$.

4.2. **Kinetic Energy of 2-D flow.** The Theorems in the last section hold for solutions to Navier-Stokes/Euler equation that decay sufficiently rapidly as $\mathbf{x} \to \infty$ so that velocity $u(.,t) \in L_2$. This is a reasonable physical assumption in $\mathbb{R}^3$.

For 2-D flow, this is not necessarily the case, unless the integral of vorticity in the flow is zero, as will be seen shortly. Suppose
$$supp\ \omega \subset \left\{ x : x \in \mathbb{R}^2, |x| < R \right\}$$
Applying 2-D Biot-Savart Law:
(4.38)
$$u(x,t) = \int_{|y| \leq R} K(x-y) \omega(y,t) dy \ , \text{ where } K(x) = \frac{1}{2\pi |x|^2} [-x_2, x_1]$$
We first note that

(4.39)

We note that

$$|x-y|^{-2} = |x|^{-2} \left(1 - 2\frac{y \cdot x}{|x|^2} + \frac{|y|^2}{|x|^2} \right) \tag{4.40}$$

Now, if $|x| \geq 2R$, then since $|y| \leq R$, it follows that as $|x| \to \infty$

$$|x-y|^{-2} = |x|^{-2} + O(|x|^{-3})$$

So, from (4.38),

$$u(x,t) = K(x) \int_{y \in \mathbb{R}^2} \omega(y,t) + O(|x|^{-3}) \tag{4.41}$$

Since

$$\int_{x \in \mathbb{R}^2} (1+|x|)^{-l} dx < \infty, \text{ iff } l > N$$

It follows that

Lemma 4.5. *A 2-D incompressible flow with compact vorticity ω has finite energy iff*

$$\int_{\mathbb{R}^2} \omega(x) dx = 0 \tag{4.42}$$

Remark 4.6. *Note that the vorticity $\omega(x,t)$ will satisfy (4.42) for $t > 0$, if*

$$\int_{\mathbb{R}^2} \omega(x,0) dx = 0, \tag{4.43}$$

since integration of 2-D Navier-Stokes equation in the vorticity form gives

$$\frac{d}{dt} \int_{x \in \mathbb{R}^2} \omega(x,t) dx = 0 \tag{4.44}$$

Remark 4.7. *The statement that finite energy is implied only iff (4.43) is satisfied is not limited merely to flow with compact support. It is more generally true for $\omega \in L_1(\mathbb{R}^2)$.*

When (4.43) is violated, it is possible to decompose a solution to Navier-Stokes equation to such that a part of it is in $L_2(\mathbb{R}^2)$ (hence finite energy), while the other part is generated by a radial distribution of vorticity whose integral is the same as the integral of initial vorticity over $\mathbb{R}^2$.

Consider an initial vorticity distribution $\omega_0(x) \in \mathbb{L}^1(\mathbb{R}^2)$. We chose any compact radial vorticity distribution $\tilde{\omega}_0(|x|)$ such that

$$\int_{\mathbb{R}^2} \tilde{\omega}_0(|x|) dx = \int_{\mathbb{R}^2} \omega_0(x) dx$$

We determine radial vorticity solution $\tilde{\omega}(|x|,t)$ with initial value $\tilde{\omega}(|x|,0) = \tilde{\omega}_0(|x|)$ to Navier-Stokes equation without forcing. We know from worked out problems two weeks back, that $\tilde{\omega}(|x|,t)$ satisfies 2-D heat equation with corresponding velocity

$$\tilde{u}(x,t) = \frac{(-x_2, x_1)}{|x|^2} \int_0^{|x|} s\tilde{\omega}_0(s,t) ds \tag{4.45}$$

Therefore, we now consider the decomposition

$$u(x,t) = \tilde{u}(x,t) + v(x,t) \tag{4.46}$$

Since $(\nabla \times v)(x,0) = \omega_0(x) - \tilde{\omega}_0(x)$, it follows that

$$\int (\nabla \times v)(x,0) dx = 0 \tag{4.47}$$

from construction of $\tilde{\omega}_0$. From (4.44) and the fact that heat solution preserves $\int_{\mathbb{R}^2} \tilde{\omega}(x,t)dx$, it follows that

(4.48)

$$\frac{d}{dt} \int_{\mathbb{R}^2} (\nabla \times v)(x,t) dx = 0 \text{ , implying by above } \int_{x \in \mathbb{R}^2} (\nabla \times v)(x,t) dx = 0$$

This implies that $v(x,t)$ has finite energy.

Thus, we have proved the following Lemma:

Lemma 4.8. *Any smooth solution $u(x,t)$ to 2-D Navier-Stokes equation with an initial $L_1(\mathbb{R}^2)$ vorticity can be decomposed into*

$$u(x,t) = v(x,t) + \tilde{u}(x,t) \tag{4.49}$$

where $v \in L_2(\mathbb{R}^2)$ and divergence free, while

$$\tilde{u}(x) = (-x_2, x_1)|x|^{-2} \int_0^{|x|} s\tilde{\omega}(s,t) ds \tag{4.50}$$

for some smooth radial vorticity distribution $\tilde{\omega}(|x|,t)$ with an initial compact support.

4.3. **Energy Inequality for 2-D flow.** Consider the *radial-Energy* decomposition

$$u(x,t) = \tilde{u}(x,t) + v(x,t) \tag{4.51}$$

of solution to the Navier-Stokes equation where $v \in L_2(\mathbb{R}^2)$. v satisfies

$$\partial_t v + v \cdot \nabla v + \tilde{u} \cdot \nabla v + v \cdot \nabla \tilde{u} = -\nabla p + \nu \Delta v + F \tag{4.52}$$

Consider two solutions to Navier-Stokes equation u_1, u_2 with radial decompositions:

$$u_1 = \tilde{u}_1 + v_1 \ , \ u_2 = \tilde{u}_2 + v_2 \tag{4.53}$$

Then, if we denote

$$w = v_1 - v_2 \ , \ \tilde{u}_1 - \tilde{u}_2 = \hat{u} \ , \ \hat{F} = F_1 - F_2 \ , \ \hat{p} = p_1 - p_2 \ , \tag{4.54}$$

then w satisfies

(4.55)

$$\partial_t w+v_1\cdot\nabla w+w\cdot\nabla v_2+\tilde{u}_1\cdot\nabla w+\hat{u}\cdot\nabla v_2+v_2\cdot\nabla\hat{u}+w\cdot\nabla\tilde{u}_1=-\nabla\hat{p}+\nu\Delta w+\hat{F}$$

Then using the same integration by parts procedure as in the last section, we have

$$\frac{d}{dt}\frac{1}{2}\|w\|_0^2+\nu\|\nabla w\|_0^2\le\|w\|_0\left\{\|w\|_0\left(\|\nabla v_2\|_\infty+\|\nabla\tilde{u}_1\|_\infty\right)\right. \\ \left.+\|\nabla(\tilde{u}_1-\tilde{u}_2)\|_\infty\|v_2\|_0+\|\hat{F}\|_0+|\tilde{u}_1-\tilde{u}_2\|_\infty\|\nabla v_2\|_0\right\} \tag{4.56}$$

Using Gronwall's inequality, as in previous section, we end up with the following proposition

Proposition 4.9. *2-D Energy Estimate and Gradient Control Let u_1 and u_2 be two smooth divergence free solutions to the Navier-Stokes equation with radial-energy decomposition $u_j(x,t)=v_j(x,t)+\tilde{u}_j(x,t)$ and with external forces F_1 and F_2. Then we have the following estimates:*

(4.57)

$$\sup_{t\in[0,T]}\|v_1-v_2\|_0\le\exp\left[\int_0^T\left(\|\nabla v_2\|_\infty+\|\nabla\tilde{u}_1\|_\infty\right)dt\right]\left\{\|v_1(.,0)-v_2(.,0)\|_0\right. \\ \left.\int_0^T\left[\|(F_1-F_2)(.,t)\|_0+\|\tilde{u}_1-\tilde{u}_2\|_\infty\|\nabla v_2\|_0+\|\nabla\tilde{u}_1-\nabla\tilde{u}_2\|_\infty\|v_2\|_0\right]dt\right\}$$

$$\nu\int_0^T\|\nabla(v_1(.,t)-v_2(.,t))\|_0^2dt\le C(v_2,\tilde{u}_1,T)\left\{\|(u_1-u_2)(.,0)\|_0^2\right. \\ \left.+\left[\int_0^T\left(\|F_1(.,t)-F_2(.,t)\|_0+\|\tilde{u}_1-\tilde{u}_2\|_\infty\|\nabla v_2(.,t)\|_0+\|\nabla\tilde{u}_1-\nabla\tilde{u}_2\|_\infty\|v_2(.,t)\|_0\right)dt\right]^2\right\} \tag{4.58}$$

Exercise: Derive (4.57) and (4.58) and use it to prove the above proposition.