Week 6 Notes, Math 8610, Tanveer

1. Global Existence Theorem for regularized Navier-Stokes

Instead of the usual Navier-Stokes equation, we consider the *regularized* Navier-Stokes equation for $u^\epsilon(x,t)$:

(1.1)
$$u_t^\epsilon + \mathcal{I}_\epsilon \left([\mathcal{I}_\epsilon u^\epsilon \cdot \nabla] \mathcal{I}_\epsilon u^\epsilon \right) = -\nabla p^\epsilon + \nu \mathcal{I}_\epsilon \mathcal{I}_\epsilon \Delta u^\epsilon \ , \ \nabla \cdot u^\epsilon = 0 \ , \ u^\epsilon(x,0) = u_0(x)$$

Using Hodge Projection operator, we project (1.1) into the space

$$V_s \equiv \left\{ v : v \in H_s(\mathbb{R}^N) \ , \ \nabla \cdot v = 0 \right\} \tag{1.2}$$

It is easily proved that the subspace V_s of H_s is itself a Banach space. Since $\mathcal{P}$ commutes with operators $\mathcal{I}_\epsilon$ and D, it follows from (1.1) that

$$u_t^\epsilon + \mathcal{P} \left\{ \mathcal{I}_\epsilon \left([\mathcal{I}_\epsilon u^\epsilon \cdot \nabla] \mathcal{I}_\epsilon u^\epsilon \right) \right\} = \nu \mathcal{I}_\epsilon \mathcal{I}_\epsilon \Delta u^\epsilon \tag{1.3}$$

This regularized Navier-Stokes equation reduces to an ODE in the Banach space V^s and can be written symbolically in the form

$$\frac{d}{dt} u^\epsilon = F_\epsilon \left(u^\epsilon \right) \ , \ u^\epsilon |_{t=0} = u_0 \tag{1.4}$$

where

$$F_\epsilon \left(u^\epsilon \right) = \nu \mathcal{I}_\epsilon^2 \Delta u^\epsilon - \mathcal{P} \left\{ \mathcal{I}_\epsilon \left([\mathcal{I}_\epsilon u^\epsilon \cdot \nabla] \mathcal{I}_\epsilon u^\epsilon \right) \right\} \equiv F_\epsilon^1(u^\epsilon) - F_\epsilon^2(u^\epsilon) \tag{1.5}$$

Lemma 1.1. *Picard Theorem in Banach Space*
Let $\mathbf{O} \subset \mathbf{B}$ *be an open set in a Banach space and* $F : \mathbf{O} \to \mathbf{B}$ *be a mapping that satisfies the following properties:*
i. F *maps* $\mathbf{O}$ *to* $\mathbf{B}$,
ii. F *is locally Lipschitz continuous, i.e. for any* $X \in \mathbf{O}$, *there exists* $L > 0$ *and on open neighborhood* $U \subset \mathbf{O}$ *containing* X *so that*

$$\| F(X_1) - F(X_2) \| \le L \| X_1 - X_2 \| \ , \ \text{for all } X_1, X_2 \in U$$

Then, for any $X_0 \in \mathbf{O}$, *there exists time* T *such that the ODE*

$$\frac{dX}{dt} = F(X) \ , \ X|_{t=0} = X_0$$

has a locally unique solution $X \in \mathbf{C}^1 \left[(-T, T) : \mathbf{O} \right]$.

Remark 1.2. *In the preceding Lemma,* $\|.\|$ *denotes the norm in the Banach space* $\mathbf{B}$.

Remark 1.3. *The proof of Lemma 1.1 is just like the classical Picard Theorem for ODEs in* $\mathbb{R}^N$; *only that* $\mathbb{R}^N$ *is replaced by Banach space* $\mathbf{B}$.

Recall that the classical Picard Theorm is based on contraction mapping theorem applied to the integral equation:

$$X(t) = X_0 + \int_0^t F(X(\tau)) d\tau \tag{1.6}$$

Smallness of T together with Lipschitz property guarantees a unique $\mathbf{C}^0\left[(-T,T),\mathbf{O}\right]$ solution. The differential equation immediately implies that this solution is also in $\mathbf{C}^1[(-T,T),\mathbf{O}]$.

Remark 1.4. *The Lemma above only guarantees local existence in t,* **the existence time T depending on the Lipschitz constant in a ball containing initial condition. This is deduced easily by applying a contraction mapping argument on (1.6)**. *To get global existence, the following Lemma is useful.*

Remark 1.5. *We will now show that each of F_1^ϵ and F_2^ϵ satisfies the conditions for applying Lemma 1.2 in the Banach space V^s for any fixed $\epsilon > 0$. By appropriately choosing an open set $\mathbf{O} \subset V^m$, we will use Lemma 1.3 to establish global existence as well.*

Lemma 1.6. *local existence for regularized problem*
For $\mathbf{O} \equiv \{u \in V^m, \|u\|_m < M\}$, the function F_ϵ defined in (1.5) satisfies the requirement that for any $u_1, u_2 \in \mathbf{O}$,

$$\|F_\epsilon(u_1) - F_\epsilon(u_2)\|_m \le c_M(\epsilon, m, N)\|u_1 - u_2\|_m$$

where constant c_M only depends on M, m, ϵ and N. Thus, F^ϵ is locally Lipschitz in $\mathbf{O}$.

Proof. Consider first $F_\epsilon^1(u_1) - F_\epsilon^1(u_2)$:

$$\|F_\epsilon^1(u_1) - F_\epsilon^1(u_2)\|_m = \nu\|\mathcal{I}_\epsilon^2 \Delta(u_1 - u_2)\|_m \le \nu\|\mathcal{I}_\epsilon{}^2(u_1 - u_2)\|_{m+2} \le \frac{c\nu}{\epsilon^2}\|(u_1 - u_2)\|_m, \tag{1.7}$$

where we used Lemma 2.4 of week 5 notes (parts iv and v). Now,

$$\begin{aligned}\|F_\epsilon^2(u_1) - F_\epsilon^2(u_2)\|_m &\le \|\mathcal{P}\left\{\mathcal{I}_\epsilon\left([\mathcal{I}_\epsilon u_1^\epsilon \cdot \nabla]\mathcal{I}_\epsilon\{u_1^\epsilon - u_2^\epsilon\}\right)\right\}\|_m + \|\mathcal{P}\left\{\mathcal{I}_\epsilon\left([\mathcal{I}_\epsilon\{u_1^\epsilon - u_2^\epsilon\} \cdot \nabla]\mathcal{I}_\epsilon u_2^\epsilon\right)\right\}\|_m \\ &\le \|\mathcal{I}_\epsilon u_1^\epsilon\|_\infty \|\mathcal{I}_\epsilon\{Du_1^\epsilon - Du_2^\epsilon\}\|_m + \|\mathcal{I}_\epsilon\{u_1^\epsilon - u_2^\epsilon\}\|_\infty \|\mathcal{I}_\epsilon D u_2^\epsilon\|_m \\ &\le c\left(\epsilon^{-N/2-1}\|u_1^\epsilon\|_0 \|u_1^\epsilon - u_2^\epsilon\|_m + \epsilon^{-N/2-m-1}\|u_1^\epsilon - u_2^\epsilon\|_0 \|u_2^\epsilon\|_0\right) \\ &\le \frac{c}{\epsilon^{N/2+1+m}}\left(\|u_1^\epsilon\|_0 + \|u_2^\epsilon\|_0\right)\|u_1^\epsilon - u_2^\epsilon\|_m\end{aligned} \tag{1.8}$$

∎

Remark 1.7. *Note that (1.7) and (1.8) implies* **that the Lipschitz constant C only depends on N, m and L_2 norm of initial $\|u^\epsilon\|_0$, but otherwise independent of $\|u^\epsilon\|_m$. In the ensuing, we will show $\|u^\epsilon(.,t)\|_0 \leq \|u_0\|_0$, and hence Lipschitz constant is independent of solution.** *Also, note that using $u_2^\epsilon = 0$ and $u_1^\epsilon = u^\epsilon$:*

$$\|F^\epsilon(u^\epsilon)\|_m \leq C(\|u^\epsilon\|_0, \epsilon, N, m)\|u^\epsilon\|_m \tag{1.9}$$

Proposition 1.8. *Consider any initial condition $u_0 \in V^m$, $m \in \mathbb{Z}^+ \cup \{0\}$. Then for any $\epsilon > 0$, there exists a unique solution $u^\epsilon \in \mathbf{C}^1\left([0, T_\epsilon]; V^m\right)$ to (1.4), where $T_\epsilon = T(\|u_0\|_m, \epsilon)$. On any time interval $[0, T]$ for which the solution belongs to $\mathbf{C}^1\left([0, T]; V^0\right)$,*

$$\sup_{0 \leq t \leq T} \|u^\epsilon\|_0 \leq \|u_0\|_0$$

Proof. Choose $\mathbf{O} \subset V^m$ a ball of radius M that contains $u = u_0$. From Lemma 1.6, it follows that F_ϵ is locally Lipschitz in M, and therefore from Picard Theorem Lemma 1.1, there exists sufficiently small $T_\epsilon > 0$, depending on $\|u_0\|_m$ and ϵ, so that there exists a unique solution $u^\epsilon \in \mathbf{C}^1\left([0, T_\epsilon], \mathbf{O}\right)$ to (1.4). This is the only solution in $\mathbf{C}^1\left([0, T_\epsilon], V^m\right)$ since for sufficiently small T_ϵ continuity implies that $\|u^\epsilon - u_0\|_m$ is small.

To show the second part of the Theorem, we note that on taking the L_2 inner product of (1.4) with u^ϵ, we obtain on using properties of mollifiers and projections (see Lemma 1.13 of week 5 notes)

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_0^2 &= \nu\left(u^\epsilon, \mathcal{I}_\epsilon^2 \Delta u^\epsilon\right)_0 - \left(u^\epsilon, \mathcal{P}\mathcal{I}_\epsilon\left[(\{\mathcal{I}_\epsilon u^\epsilon\} \cdot \nabla)(\mathcal{I}_\epsilon u^\epsilon)\right]\right)_0 \\ &= -\nu\left(\mathcal{I}_\epsilon \nabla u^\epsilon, \mathcal{I}_\epsilon \nabla u^\epsilon\right) - \left(\mathcal{I}^\epsilon u^\epsilon, (\{\mathcal{I}_\epsilon u^\epsilon\} \cdot \nabla)(\mathcal{I}_\epsilon \nabla u^\epsilon)\right)_0 \end{aligned}$$

Now since $v^\epsilon \equiv \mathcal{I}^\epsilon u^\epsilon$ is divergence free, it follows that

$$\left(v^\epsilon, (v^\epsilon \cdot \nabla) v^\epsilon\right)_0 = 0$$

just as in the usual Navier-Stokes equation. So,

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_0^2 + \nu\|\mathcal{I}_\epsilon D u^\epsilon\|_0^2 = 0$$

Therefore,

$$\|u^\epsilon\|_0^2 \leq \|u_0\|_0^2$$

and the second Lemma statement follows. ∎

Theorem 1.1. *Global Existence for regularized N-S equation*

For any $T > 0$ and initial condition $u_0 \in V_m$, the regularized Navier Stokes equation (1.4) has a solution $u^\epsilon \in \mathbf{C}^1\left([0, T], V_m\right)$.

Proof. First, we note from (1.4), (1.9) that

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_m^2 = (u^\epsilon, \partial_t u^\epsilon)_m = (u^\epsilon, F^\epsilon(u^\epsilon))_m \leq C(\|u_0\|_0, \epsilon)\|u^\epsilon\|_m^2$$

Therefore,

$$\|u^\epsilon(.,t)\|_m \leq \|u_0\|_m e^{Ct}$$

For any $T > 0$, choose

$$\mathbf{O} \equiv \left\{ u^\epsilon : u^\epsilon \in V^m, \|u^\epsilon\|_m < 2\|u_0\|_m e^{CT} \right\}$$

We know local $\mathbf{C}^1\left([0, T_\epsilon], \mathbf{O}\right)$ solution exists from previous proposition (1.8), **where T_ϵ only depends on ϵ, m and $\|u_0\|_0$, but otherwise independent of $\|u_0\|_m$. This is because the Lipschitz constant as pointed out in Remark 1.7 is only dependent on $\|u_0\|_0$, ϵ and m. Since $\|u(., T_\epsilon)\|_0 \leq \|u_0\|_0$, we may restart the clock at T_ϵ and continue in steps of T_ϵ until we get to $t = T$.** ∎

Remark 1.9. *Though the solution to the regularized Navier-Stokes equation (1.4) exists for all time, going to the limit $\epsilon \to 0$ is not possible with the energy bounds obtained so far because they depend badly on ϵ. So, now we seek energy bounds independent of ϵ; this will be possible only locally in time, as shall be seen shortly., Nonetheless, this allows us one to take $\epsilon \to 0$ and obtain actual solution of Navier-Stokes equation locally in time.*

Lemma 1.10. *ϵ independent Energy bounds for regularized problem:*
Let $u_0 \in V^m$. Then the unique solution $u^\epsilon \in \mathbf{C}^1\left([0,\infty); V^m\right)$ to the regularized Navier-Stokes equation guaranteed by Theorem 1.1 satisfies the following inequality

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_m^2 + \nu\|\mathcal{I}_\epsilon \nabla u^\epsilon\|_m^2 \leq c_m \|\nabla \mathcal{I}_\epsilon u^\epsilon\|_\infty \|u^\epsilon\|_m^2$$

Further, for $m > N/2 + 1$, we obtain for sufficiently small T,

$$\sup_{t\in[0,T]} \|u^\epsilon\|_m \leq \frac{\|u_0\|_m}{1 - c_m T\|u_0\|_m} = \|u_0\|_m + \frac{\|u_0\|_m^2 c_m T}{1 - c_m T\|u_0\|_m}$$

Proof. We note that for any α, with $|\alpha| \leq m$,
(1.10)

$$\left(D^\alpha u^\epsilon, \partial_t D^\alpha u^\epsilon\right)_0 = \left(D^\alpha u^\epsilon, D^\alpha \mathcal{I}_\epsilon^2 \Delta u^\epsilon\right)_0 - \left(D^\alpha u^\epsilon, D^\alpha \mathcal{P}\left\{\mathcal{I}_\epsilon\left([\mathcal{I}_\epsilon u^\epsilon \cdot \nabla]\mathcal{I}_\epsilon u^\epsilon\right)\right\}\right)_0$$

However, it is clear from properties of $\mathcal{I}_\epsilon$ that

$$\left(D^\alpha u^\epsilon, D^\alpha \mathcal{I}_\epsilon^2 \Delta u^\epsilon\right)_0 = -\left(D^\alpha \nabla \mathcal{I}^\epsilon u^\epsilon, D^\alpha \nabla \mathcal{I}_\epsilon^2 u^\epsilon\right)_0$$

Further, on defining $v^\epsilon = \mathcal{I}_\epsilon u^\epsilon$, we get

(1.11)
$$\begin{aligned}(D^\alpha u^\epsilon, D^\alpha \mathcal{P}\left\{\mathcal{I}_\epsilon\left([\mathcal{I}_\epsilon u^\epsilon \cdot \nabla]\mathcal{I}_\epsilon u^\epsilon\right)\right\})_0 &= \left(D^\alpha \mathcal{I}_\epsilon u^\epsilon, D^\alpha\left[(\mathcal{I}_\epsilon u^\epsilon \cdot \nabla)\mathcal{I}_\epsilon u^\epsilon\right]\right)_0 \\ &= \left(D^\alpha v^\epsilon, D^\alpha\left[(v^\epsilon \cdot \nabla)v^\epsilon\right] - (v^\epsilon \cdot \nabla)D^\alpha v^\epsilon\right)_0,\end{aligned}$$

since for any divergence free vector field v^ϵ, $(w, v^\epsilon \cdot \nabla w) = 0$. However, taking $w^\epsilon = D^\alpha v^\epsilon$, we obtain from using Lemma 1.12, week 5 lecture notes:

$$\left|\left(D^\alpha v^\epsilon, D^\alpha\left[(v^\epsilon \cdot \nabla)v^\epsilon\right] - (v^\epsilon \cdot \nabla)D^\alpha v^\epsilon\right)_0\right| \le \|Dv^\epsilon\|_\infty \|D^\alpha v^\epsilon\|_0^2$$

for $m > N/2+1$. Therefore, it follows from (1.10)-(1.11) summing over α, with $|\alpha| \le m$, we obtain

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_m^2 + \nu\|\mathcal{I}_\epsilon \nabla u^\epsilon\|_m^2 \le c_m \|\nabla \mathcal{I}_\epsilon u^\epsilon\|_\infty \|u^\epsilon\|_m^2$$

Now, for $m > N/2+1$,

$$\|\nabla v^\epsilon\|_\infty \le c\|v^\epsilon\|_m \le c\|u^\epsilon\|_m$$

Therefore,

$$\frac{d\|u^\epsilon\|_m}{dt} \le c_m \|u^\epsilon\|_m^2$$

Integration gives rise to the desired energy bounds. ∎

2. Local Existence for Navier-Stokes equation

We now use the ϵ-independent energy bounds for solutions to mollified Navier-Stokes equation to prove local existence of solution for the actual Navier-Stokes equation. First, we show that it forms a Cauchy sequence in an appropriate space:

Lemma 2.1. *For $m > N/2+2$, consider the family $\{u^\epsilon\}_\epsilon$ of solution to the regularized N-S equation with same initial condition $u^\epsilon(.,0) = u_0 \in V^m(\mathbb{R}^N)$ over time interval $[0,T]$, where $T < \frac{1}{c_m\|u_0\|_m}$. Note that we have ϵ-independent energy bounds on this time interval. This forms a Cauchy sequence in $\mathbf{C}\left\{[0,T], \mathbf{L}^2(R^3)\right\}$. Further, there exists a constant C only depending on $\|u_0\|_m$ and time T so that for all $\epsilon \ge \epsilon' > 0$,*

$$\sup_{t\in[0,T]} \|u^\epsilon - u^{\epsilon'}\|_0 \le C\epsilon$$

Proof. Using $\frac{d}{dt}u^\epsilon = F_\epsilon(u^\epsilon)$ for $\epsilon = \epsilon$ and $\epsilon = \epsilon'$, subtracting the equation and taking the inner-product in L^2, we obtain

$$(2.12)\quad \frac{d}{dt}\frac{1}{2}\|u^{\epsilon'} - u^\epsilon\|_0^2 = \nu\left(\mathcal{I}_{\epsilon'}^2\Delta u^{\epsilon'} - \mathcal{I}_\epsilon^2\Delta u^\epsilon, u^{\epsilon'} - u^\epsilon\right)$$
$$- \left(\mathcal{P}\mathcal{I}_{\epsilon'}\left[\mathcal{I}_{\epsilon'}u^{\epsilon'}\cdot\nabla\mathcal{I}_{\epsilon'}u^{\epsilon'} - \mathcal{I}_\epsilon u^\epsilon\cdot\nabla\mathcal{I}_\epsilon u^\epsilon\right], u^{\epsilon'} - u^\epsilon\right) \equiv T1 + T2$$

We first estimate T_1:

$$(2.13)\quad T_1 = \nu\left(\left\{\mathcal{I}_{\epsilon'}^2 - \mathcal{I}_\epsilon^2\right\}\Delta u^{\epsilon'}, u^\epsilon - u^{\epsilon'}\right) + \nu\left(\mathcal{I}_\epsilon^2\Delta[u^{\epsilon'} - u^\epsilon], u^{\epsilon'} - u^\epsilon\right)$$

Using part (iv) of Lemma 1.13 of week 5 notes, and taking $w = \Delta u^{\epsilon'}$, we obtain

$$\|\mathcal{I}_{\epsilon'}^2 w - \mathcal{I}_\epsilon^2 w\| \le \|\mathcal{I}_{\epsilon'}^2 w - \mathcal{I}_{\epsilon'}w\| + \|\mathcal{I}_{\epsilon'}w - w\| + \|\mathcal{I}_\epsilon^2 w - \mathcal{I}_\epsilon w\| + \|\mathcal{I}_{\epsilon'}w - w\|_0 \le C\epsilon\|w\|_1$$

Therefore, using above and integration by parts on the latter term in T_1, we obtain

$$(2.14)\qquad |T_1| \le C\nu\epsilon\|u^\epsilon\|_3\|u^{\epsilon'} - u^\epsilon\|_0 - \nu\|\mathcal{I}_\epsilon\nabla(u^{\epsilon'} - u^\epsilon)\|_0^2$$

Now, with respect to T_2, it is convenient to decompose

$$T_2 = \left(\mathcal{P}(\mathcal{I}_{\epsilon'} - \mathcal{I}_\epsilon)\left[\mathcal{I}_{\epsilon'}u^{\epsilon'}\cdot\nabla\mathcal{I}_{\epsilon'}u^{\epsilon'}\right], u^{\epsilon'} - u^\epsilon\right) + \left(\mathcal{P}\mathcal{I}_\epsilon\left[(\mathcal{I}_{\epsilon'} - \mathcal{I}_\epsilon)u^{\epsilon'}\cdot\nabla\mathcal{I}_{\epsilon'}u^{\epsilon'}\right], u^{\epsilon'} - u^\epsilon\right) +$$
$$+\left(\mathcal{P}\mathcal{I}_\epsilon\left[\mathcal{I}_\epsilon(u^{\epsilon'} - u^\epsilon)\cdot\nabla\mathcal{I}_{\epsilon'}u^{\epsilon'}\right], u^{\epsilon'} - u^\epsilon\right) + \left(\mathcal{P}\mathcal{I}_\epsilon\left[\mathcal{I}_\epsilon u^\epsilon\cdot\nabla(\mathcal{I}_{\epsilon'} - \mathcal{I}_\epsilon)u^{\epsilon'}\right], u^{\epsilon'} - u^\epsilon\right)$$
$$+\left(\mathcal{P}\mathcal{I}_\epsilon\left[\mathcal{I}_\epsilon u^\epsilon\cdot\nabla\mathcal{I}_\epsilon(u^{\epsilon'} - u^\epsilon)\right], u^{\epsilon'} - u^\epsilon\right) \equiv T_{2,1} + T_{2,2} + T_{2,3} + T_{2,4} + T_{2,5}$$

Now, we note that for some C, independent of ϵ,

$$|T_{2,1}| \le C\epsilon\|u^{\epsilon'}\|_1\|u^{\epsilon'} - u^\epsilon\|_0\|\mathcal{I}_\epsilon'\nabla u^{\epsilon'}\|_\infty \le C\epsilon\|u^{\epsilon'}\|_m^2\|u^\epsilon - u^{\epsilon'}\|_0$$
$$|T_{2,2}| \le C\epsilon\|u^{\epsilon'}\|_1\|u^{\epsilon'} - u^\epsilon\|_0\|\mathcal{I}_{\epsilon'}\nabla u^{\epsilon'}\|_\infty \le C\epsilon\|u^{\epsilon'}\|_m^2\|u^\epsilon - u^{\epsilon'}\|_0$$
$$|T_{2,3}| \le C\|u^{\epsilon'} - u^\epsilon\|_0^2\|\mathcal{I}_{\epsilon'}\nabla u^{\epsilon'}\|_\infty \le C\|u^{\epsilon'}\|_m\|u^\epsilon - u^{\epsilon'}\|_0^2$$
$$|T_{2,4}| \le C\epsilon\|u^{\epsilon'}\|_1\|u^{\epsilon'} - u^\epsilon\|_0\|\mathcal{I}_{\epsilon'}\nabla u^{\epsilon'}\|_\infty \le C\epsilon\|u^{\epsilon'}\|_m^2\|u^\epsilon - u^{\epsilon'}\|_0$$

For $T_{2,5}$, it is useful to subsitute $v = \mathcal{I}_\epsilon u^\epsilon$, $w = \mathcal{I}_\epsilon(u^{\epsilon'} - u^\epsilon)$. Note that w and v is divergence free. Then we note that

$$T_{2,5} = (v\cdot\nabla w, w) = \int_{x\in\mathbb{R}^N} w_i v_j \partial_{x_j} w_i = 0$$

Therefore, from (2.12) and previous ϵ independent bound on $\|u^\epsilon\|_m$ over an interval $[0, T]$, (in last week's notes), it follows that

$$\frac{d}{dt}\|u^{\epsilon'} - u^\epsilon\|_0 \le C_m(T)\left(\epsilon + \|u^{\epsilon'} - u^\epsilon\|_0\right)$$

Using Gronwall's inequality, it follows that there exists some constant C depending on T so that for any $t \in [0,T]$,

$$\|u^{\epsilon'}(.,t) - u^{\epsilon}(.,t)\|_0 \leq \epsilon C$$

∎

Proposition 2.2. *If initial condition $u_0 \in V^m$ for $m > N/2+2$, then for $T < \frac{1}{c_m \|u_0\|_m}$, there exists a solution to Navier-Stokes equation $u \in \mathbf{C}\left([0,T], V^{m'}(\mathbb{R}^N)\right)$, while $\partial_t u \in \mathbf{C}\left([0,T], V^{m'-2}(\mathbb{R}^N)\right)$ for any $N/2+2 < m' < m$. More over, this solution is classical in the sense that $u \in \mathbf{C}^0\left([0,T], \mathbf{C}^2(\mathbb{R}^N)\right)$, $\partial_t u \in \mathbf{C}^0\left([0,T], \mathbf{C}(\mathbb{R}^N)\right)$.*

Proof. Assume without loss of generality that $\epsilon' \leq \epsilon$. We note that for $t \in [0,T]$, $\|v^\epsilon\|_m \leq C$, independent of ϵ. From interpolation inquality for Sobolev norms and Lemma (2.12), for any $t \in [0,T]$,

$$\|u^\epsilon(.,t) - u^{\epsilon'}(.,t)\|_{m'} \leq c\|u^\epsilon(.,t) - u^{\epsilon'}(.,t)\|_0^{1-m'/m} \|u^\epsilon(.,t) - u^{\epsilon'}(.,t)\|_m^{m'/m} \leq C_m(T)\epsilon^{1-m'/m}$$

Thus, u^ϵ forms a Cauchy sequence in $\mathbf{C}^0\left([0,T], V^{m'}(\mathbb{R}^N)\right)$ and hence converges to a function u in the same space. Since $m' > N/2+2$, it follows that $u \in \mathbf{C}^0\left([0,T], \mathbf{C}^2(\mathbb{R}^N)\right)$. Further, by taking the limit of $\epsilon \to 0$ it follows that

$$\lim_{\epsilon \to 0^+} \nu \left(\mathcal{I}_\epsilon^2 \Delta u^\epsilon - \mathcal{P}\mathcal{I}_\epsilon \left[\mathcal{I}_\epsilon u^\epsilon \cdot \nabla \mathcal{I}_\epsilon u^\epsilon\right]\right) = \nu \Delta u - \mathcal{P}[u \cdot \nabla u]$$

Therefore,

$$\lim_{\epsilon \to 0} u_t^\epsilon = \nu \Delta u - \mathcal{P}[u \cdot \nabla u]$$

and the limiting function satisfies Navier-Stokes equation. Since $\lim_{\epsilon \to 0} u^\epsilon = u$ in $\mathbf{C}^0\left([0,T], V^{m'}(\mathbb{R}^N)\right)$, it follows that at least in the sense of distribution, we have $\lim_{\epsilon \to 0^+} u_t^\epsilon = u_t$. Therefore, the limiting function u satisfies the Navier-Stokes equation and satisfies initial condition u_0. From the equation itself, it follows that we have $u_t \in \mathbf{C}^0\left([0,T], V^{m'-2}(\mathbb{R}^N)\right)$.
∎

Remark 2.3. *The above proposition is not completely satisfactory since it suggests that if $u_0 \in V^m$, then it only assures $u(.,t) \in V^{m'}$, for $m' < m$. In reality $u(.,t) \in V^m$ as well. However, to show this we need to work a bit harder.*

Definition 2.4. *A sequence $\{v_n\}_n$ in a Hilbert Space $\mathcal{H}$ is said to converge weakly to v if for any $w \in \mathcal{H}$, $\lim_{n\to\infty}(w, v_n) = (w, v)$.*

A property of weakly convergent sequence that will be important for us is that they are also bounded. Also, the following Theorem proved in any standard text in analysis is useful:

Theorem 2.1. *(Banach-Alogou Theorem) Any bounded sequence in a Hilbert Space*[(1)] *has a subsequence that converges weakly.*

Remark 2.5. *In our context, the Hilbert Space* $\mathcal{H} = V^m$.

Definition 2.6. *A function* $v(.,t) \in \mathcal{B}$*, a reflexive Banach Space, is said to be weakly continuous for* $t \in [0,T]$ *if for any* $w \in \mathcal{B}^*$*, the dual Banach space, we have* $< w, v(.,t) >$ *a continuous function of* $t \in [0,T]$.

Theorem 2.2. *Local Existence of N-S solutions Let* u *be the solution described by the previous proposition. Then*

$$v \in \mathbf{C}\left([0,T], V^m\right) \cup \mathbf{C}^1\left([0,T], V^{m-2}\right)$$

Proof. We know from prior energy estimates on the regularized Navier-Stokes equation that

$$\sup_{t\in[0,T]} \|u^\epsilon\|_m \le M$$

and from the regularized N-S equation itself, it follows that

$$\sup_{t\in[0,T]} \|u^\epsilon\|_{m-2} \le M_1$$

for some constants M and M_1. Since $\{u^\epsilon\}_{\epsilon=1/n}$ is a bounded sequence in the Hilbert Space $L^2\left([0,T], V^m\right)$. Theorem 2.1 implies that there exists a subsequence which converges to $u \in L^2\left([0,T], V^m\right)$, as $n \to \infty$ ($\epsilon \to 0$). This must be the same u as in Proposition 2.2 since $V^{m'} \subset V^m$ and $\lim_{\epsilon\to 0} u^\epsilon = u$ in $\mathbf{C}\left([0,T], V^{m'}\right)$. Further, for each $t \in [0,T]$, since u^ϵ is a bounded sequence in the Hilbert Space V^m, there is a subsquence that converges to $u(.,t) \in V^m$. Thus, it follows that

$$u \in \mathbf{L}^\infty\left([0,T], V^m\right)$$

Further, we claim

$$u \in \mathbf{C}_W\left([0,T], V^m\right)$$

First, we note that for $0 < m' < m$, the space $V^{-m'}$ is dense in V^{-m}. Hence we take arbitrary $\phi \in V^{-m'}$ and note that $< \phi, u(.,t) >$ is continuous in t for $t \in [0,T]$, because $u \in \mathbf{C}\left([0,T], V^{m'}\right)$. Therefore, the claim follows.

In view of weak continuity, we note that

$$\lim_{\delta\to 0} (u(.,t+\delta) - u(.,t), u(.,t+\delta) - u(.,t))_m = \lim_{\delta\to 0} \left(\|u(.,t+\delta)\|_m^2 - \|u(.,t)\|_m^2\right)$$

Thus to show $u \in \mathbf{C}\left([0,T], V^m\right)$, it is enough to show $\|u(.,t)\|_m$ is continuous.

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(1) More generally a reflexive Banach Space, in which case the definition of weak convergence involves the dual space

We first prove the right continuity at $t = 0$. We choose $\phi_0 \in V^{-m}$ **so that for any** $v \in V_m$**,** $< \phi_0, v >= (u_0, v)_m$**. In particular, this implies** $\|u_0\|^2 =< \phi_0, u_0 >$. Then from weak continuity,

$$\lim_{t \to 0^+} < \phi_0, u(., t) >= \|u_0\|_m^2$$

Therefore, **since** $\|u_0\|_m \|u\|_m \geq < \phi_0, u >$

$$\text{liminf}_{t \to 0^+} \|u(., t)\|_m \geq \|u_0\|_m$$

Now, from energy bounds,

$$\sup_{t \in [0,T]} \|u(., t)\|_m - \|u_0\|_m \leq \frac{\|u_0\|_m^2 c_m T}{1 - c_m T \|u_0\|_m}$$

Therefore,

$$\text{limsup}_{t \to 0^+} \|u(., t)\| \leq \|u_0\|_m$$

So, right continuity of $\|u(., t)\|_m$ has been proved at $t = 0$. It is clear that for any $t \in [0, T]$, we can repeat the same argument to show the right continuity.

To show left continuity, we have to deal differently for $\nu = 0$ (Euler Equation) and $\nu > 0$.

For $\nu = 0$, the equations are time reversible, meaning that if we replace t by $-t$ and u by $-u$, we get back the same (Euler) equation. So, left continuity follows from the same argument as the one above for right continuity.

For $\nu > 0$, we recall the energy inequality for $t \in [0, T]$:

$$\|u^\epsilon(., t)\|_m^2 + \nu \int_0^t \|\mathcal{I}_\epsilon \nabla u^\epsilon(., \tau)\|_m^2 d\tau = \|u^\epsilon(., 0)\|_m^2,$$

implying that there exists C independent of ϵ so that

$$\nu \int_0^T \|\mathcal{I}_\epsilon \nabla u^\epsilon(., t)\|_m^2 dt \leq C$$

Since $\|\mathcal{I}_\epsilon v\|_{m+1} \to \|v\|_{m+1}$ as $\epsilon \to 0^+$, it follows that $\{u^\epsilon\}_{\epsilon=1/n}$ is a bounded sequence in the Hilbert space $L^2\left([0, T], V^{m+1}\right)$. It follows that there is a subsequence that converges to u as $\epsilon \to 0$. This implies that for almost any $t \in [0, T]$, $u(., t) \in V^{m+1}$. Suppose we want to show left continuity at $t = T_1 \in [0, T]$. We choose $T_1 > T_0 > 0$ so that solution $u(., T_0) \in V^{m+1}$. Then, starting at $T = T_0$, we continue. We can apply Proposition 2.2, with initial condition $u(., T_0)$ This ensures solution in $\mathbf{C}\left([T_0, T'], V^{\tilde{m}}\right)$ for $\tilde{m} < m + 1$ for some $T' > T_0$. However, from uniqueness of classical solution, it follows that this is the same solution $u \in \mathbf{C}\left([0, T], V^{m'}\right)$ for $m' < m$ guaranteed by Proposition 2.2. Therefore, $u \in \mathbf{C}\left([T_0, T'], V^{\tilde{m}}\right)$ can be continued past T' if

$\|u(.,T')\|_{m+1} < \infty$ Indeed, T' can be extended to be as large as we like so long as $\|u(.,t)\|_{m+1}$ remains finite for $t \in [T_0, T']$.

However, in the process of derivation of ϵ independent energy bounds (see Lemma 1.10 first statement), we notice that as long as $\|\mathcal{I}_\epsilon \nabla u^\epsilon\|_\infty < C$, where C is independent of ϵ, then so is $\|u^\epsilon(.,t)\|_{m+1}$. However, $\|\mathcal{I}_\epsilon \nabla u^\epsilon(.,t)\|_\infty \leq c\|u^\epsilon(.,t)\|_m < C$, independent of ϵ for $t \in [0,T]$. Therefore, $\|u(.,t)\|_{m+1} < \infty$ for $t \in [T_0, T']$ for any $T' \leq T$. Therefore $u \in \mathbf{C}\left([T_0, T], V^{\tilde{m}}\right)$ for any $\tilde{m} < m+1$ and in particular for $\tilde{m} = m$. Hence the left continuity of $\|u(.,t)\|_m$ at $T = T_1$ follows. ∎

3. Sufficient Condition for Global Existence for N-S solution

First, we show that local unique NS solution in $C\left([0,T], V_m\right)$ for $m > \frac{N}{2} + 2$ that was proved in the last section may be extended beyond T (Recall $T < \frac{1}{c_m \|u_0\|_m}$) as long as $\|u(.,t)\|_m$ remains finite.

Lemma 3.1. *Assume $[0, \tilde{T})$ is the largest interval for which NS solution $u \in C\left([0,\tilde{T}), V_m\right)$ for $m > \frac{N}{2} + 2$ exists. If $\tilde{T} < \infty$, then $\|u(.,t)\|_m$ blows up as $t \to \tilde{T}^-$.*

Proof. Assume otherwise; therefore, $\sup_{t \in [0,\tilde{T})} \|u(.,t)\|_m \leq M < \infty$. We know that if we restart the clock at any $t_0 \in [0, \tilde{T})$, solution will exist over a time interval $[t_0, t_0 + T]$ for any $T < \frac{1}{c_m M}$. In particular, if we choose $t_0 = \tilde{T} - \frac{1}{2 c_m M}$, the interval $[0, t_0 + T]$ of existence of NS solution will exceed $[0, \tilde{T})$ contradicting the definition of $\tilde{T}$. Hence $\|u(.,t)\|_m$ cannot remain finite as $t \to \tilde{T}^-$. ∎

Corollary 3.2. *If for finite $\tilde{T}$, $[0,\tilde{T})$ is the maximal time for existence of NS solution in V_m for $m > \frac{N}{2} + 2$, then $\int_0^t \|\nabla u(.,\tau)\|_\infty d\tau$ must blow up as $t \to \tilde{T}^-$.*

Proof. This simply follows from energy inequality, which follows from the first statement of Lemma 1.10 as $\epsilon \to 0$:

$$\frac{d}{dt}\frac{1}{2}\|u(.,t)\|_m^2 \leq C_m \|\nabla u(.,t)\|_\infty \|u(.,t)\|_m^2$$

and use of Gronwall and previous Lemmas. ∎

We will now prove a sufficient condition for global existence of classical solutions to Navier-Stokes equation is the existence of L^1 in time bounds of the L^∞ space norm of the vorticity.

For that purpose we need some properties of the Biot Savart Kernel $K_N(x)$ that occurs in the relation between velocity and vorticity. Recall that in 2-D,

$$K_2(x) = \frac{1}{2\pi|x|^2}\left(-x_2, x_1\right), \tag{3.15}$$

where as in 3-D, K_3 is an operator defined by

$$K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3} \tag{3.16}$$

It is to be noted from (3.18) and (3.16) that

$$K_N(\lambda x) = \lambda^{1-N} K_N(x)\ ,\ \text{for } \lambda > 0\ ,\ 0 \neq x \notin \mathbb{R}^N \tag{3.17}$$

and hence K_N is homogeneous of degree $(1-N)$.

Definition 3.3. *The principal value integral $PV \int_{\mathbb{R}^N}$ will be defined such that*

$$PV \int_{\mathbb{R}^N} f(x)dx = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} f(x)dx$$

Lemma 3.4. *Let $K(x)$ be a function smooth outside $x = 0$ and homogeneous of degree $1 - N$. Then $\partial_{x_j} K$ in the sense of distribution is a linear functional defined by*

$$(\partial_{x_j} K, \phi)_0 = -(K, \partial_{x_j}\phi)_0 = PV \int_{\mathbb{R}^N} \partial_{x_j} K \phi dx - c_j(\delta, \phi)_0\ ,\ \text{for all}\ \ \phi \in \mathbf{C}_c^\infty,$$

where δ is the Dirac distribution and $c_j = \int_{|x|=1} x_j K(x)dx$

Proof. We note that since $K \in L^1_{loc}(\mathbb{R}^N)$, from use of dominated convergence theorem, it follows that

$$(K, \partial_{x_j}\phi)_0 = \lim_{\epsilon\to 0^+} \int_{|x|\geq\epsilon} K \partial_{x_j}\phi dx = \lim_{\epsilon\to 0^+} \left\{ -\int_{|x|\geq\epsilon} \partial_{x_j} K\phi\ dx + \int_{|x|=\epsilon} K\phi \frac{x_j}{|x|} dx \right\}$$

The first term on the right hand side gives $PV \int$. In the second term changing variable $x \to \epsilon x$ and use of homogeneous property of K gives rise to

$$\lim_{\epsilon\to 0^+} \int_{|x|=\epsilon} K\phi \frac{x_j}{|x|} dx = \lim_{\epsilon\to 0^+} \int_{|x|=1} \epsilon^{1-N} K(x)\phi(\epsilon x)\frac{x_j}{|x|}\epsilon^{N-1} dx = \phi(0)c_j$$

Hence the Lemma follows. ∎

Lemma 3.5. *Potential Theory Results*

Let u be a smooth, $L^2 \cap L^\infty$ divergence free velocity field and $\omega = \nabla \times u$. Then

$$\|\nabla u\|_\infty \leq c\left(1 + \ln^+ \|u\|_3 + \ln^+ \|\omega\|_0\right)\left(1 + \|\omega\|_\infty\right),$$

where $\ln^+ v = \ln v$ if $v > 1$ and 0 otherwise.

Remark 3.6. *The proof relies on the expression*

$$\nabla u(x) = PV \int_{\mathbb{R}^N} \nabla_x K_N(x-y)\omega(y)dy + c\omega(x) \tag{3.18}$$

Details given in Proposition 3.8 and Lemma 4.6 in Bertozzi & Majda book. This is a result from potential theory and has nothing to do with the evolution of $u(x,t)$ in Navier-Stokes equation.

Theorem 3.1. *Beale-Kato-Majda sufficient condition for global regularity*

Let initial $u_0 \in V^m$, $m > N/2 + 2$ so that there exists a classical solution u to Navier-Stokes or Euler equation, locally in time. Then, if for any $T > 0$, if there exists constant C so that

$$\int_0^T \|\omega(.,t)\|_\infty dt \leq C,$$

then, the solution to Navier-Stokes equation exists globally in time, i.e. $u \in \mathcal{C}^0([0,\infty), V^m) \cap \mathcal{C}^1([0,\infty), V^{m-2})$. Also, if the maximal time for existence $T < \infty$, then

$$\lim_{t\to T^-} \int_0^T \|\omega(.,t)\|_\infty dt = \infty$$

Proof. We have shown that

$$\int_0^T \|\nabla u(.,t)\|_\infty dt \leq C,$$

is enough to guarantee a classical solution in $[0,T]$ since

$$\|u(.,T)\|_m \leq \|u_0\|_m \exp\left[\int_0^T c_m \|\nabla u(.,t)\|_\infty dt\right]$$

So, we only need to show that $\int_0^T \|\nabla u(.,t)\|_\infty dt$ is controlled by similar integral over ω.

Since vorticity ω satisfies

$$\omega_t + u \cdot \nabla\omega = \omega \cdot \nabla u + \nu\Delta\omega$$

by taking the inner product with ω it follows that

$$\frac{d}{dt}\frac{1}{2}\|\omega(.,t)\|_0^2 \leq \|\nabla u(.,t)\|_\infty \|\omega(.,t)\|_0^2,$$

implying

$$\|\omega(.,t)\|_0 \leq \|\omega_0\|_0 \exp\left[\int_0^T \|\nabla u(.,t)\|_\infty dt\right]$$

Using above estimate on $\|u(.,t)\|_m$ for $m = 3$ and above estimate for $\|\omega(.,t)\|_0$, it follows from the potential theory estimates of 3.5 that

$$\|\nabla u(.,t)\|_\infty \leq C \left[1 + \int_0^t \nabla u(.,\tau) d\tau \right] (1 + \|\omega(.,t)\|_\infty)$$

Therefore, using Gronwall's Lemma

$$\|\nabla u(.,t)\|_\infty \leq \|\nabla u_0\|_\infty \exp \left[C \int_0^t (1 + \|\omega(.,\tau)\|_\infty)\, d\tau \right]$$

∎

Corollary 3.7. *For $N = 2$, NS solution $u(.,t) \in V_m$ exists globally in time.*

Proof. Assume otherwise, *i.e.* there exists maximal time interval $[0, T)$, for $T < \infty$. Recall in 2-D scalar ω satisfies

$$\omega_t + u \cdot \omega = \nu \Delta \omega,$$

using maximum principle, $\|\omega(.,t)\|_\infty \leq \|\omega_0\|_\infty$, implying that $\int_0^T \|\omega(.,t)\|_\infty dt$ is finite and hence from BKM, solution exists in $[0, T]$. Since $\|u(.,T)\|_m$ is finite, the solution may be extended beyond T, contradicting definition of T. ∎

Remark 3.8. *Note that the above Corollary holds for forced NS equation as well using similar arguments.*