

Week 7 Notes, Math 8610, Tanveer

1. STOKES FLOW

We now turn to flow for large viscosity ν . In this case, we have shown that at least in some geometry, we have shown that the solution to the Navier-Stokes initial value problem approaches solutions to Stokes equation, where nonlinear momentum terms in Navier-Stokes equations are neglected. This is more generally true. In the Stokes limit, we have the following equation:

$$(1.1) \quad u_t = -\nabla p + \nu \Delta u + b, \quad \nabla \cdot u = 0$$

We will spend a few lectures discussion some explicit solution to this linear equation. First, we note that if the domain $\Omega = \mathbb{R}^N$, then the initial value problem corresponding to (1.1) has explicit solution in the Fourier-Space

$$(1.2) \quad \hat{u}(k, t) = \hat{u}_0(k) e^{-|k|^2 t} + \int_0^t e^{-|k|^2(t-\tau)} \hat{b}(k, \tau) d\tau$$

We can verify in (1.2) that the divergence condition, which is equivalent in Fourier-Space to $k \cdot \hat{u} = 0$ since $k \cdot \hat{u}_0 = 0 = k \cdot \hat{b}$.

1.1. Steady Exact Solution in \mathbb{R}^3 : (Stokeslet Solution). First, we look at solutions in \mathbb{R}^3 for the steady case, where $u_t = 0$; in this case we have

$$(1.3) \quad 0 = -\nabla p + \nu \Delta u + b, \quad \nabla \cdot u = 0$$

Since we can rescale u and x , there is no loss of generality in choosing $\nu = 1$. This equation has an explicit solution representation when domain $\Omega = \mathbb{R}^3$. Indeed, since the equation is linear, there is particular interest in the Fundamental solution for which the force $b = A\delta(x)$ where $A \in \mathbb{R}^3$.

$$(1.4) \quad 0 = -\nabla p + \Delta u + A\delta(x), \quad \nabla \cdot u = 0$$

There are two ways of solving this: one involves use of Fourier Transform in \mathbb{R}^3 , solving for $\hat{u}(\mathbf{k})$ followed by inverse Fourier-Transform. The other is to introduce a spherical coordinate system, look for appropriate singular solution at $|x| = 0$, corresponding to a delta function forcing as given in (1.4). Fourier-Transform is easier in this case, but we need to understand how to solve in otherways as preparation for more complicated domains $\Omega \neq \mathbb{R}^3$:

We denote by $\hat{u}(k)$ the generalized Fourier-transform, in the sense of distribution, of $u(x)$. Recall that $\mathcal{F}[\Delta u] = -|k|^2 \hat{u}$ and that the Hodge Projection \mathcal{P} in the Fourier-Space has the following representation:

$$\mathcal{F}\{\mathcal{P}v\}(k) = \hat{v}(k) - \frac{k(k \cdot \hat{v})}{|k|^2}$$

Therefore, (1.4) implies:

$$(1.5) \quad \hat{u}(k) = \frac{1}{(2\pi)^3 |k|^2} \left[A - (k \cdot A) \frac{k}{|k|^2} \right]$$

Therefore,

$$(1.6) \quad u(x) = \frac{1}{8\pi^3} \int_{k \in \mathbb{R}^3} e^{ik \cdot x} \left[A - (k \cdot A) \frac{k}{|k|^2} \right] \frac{dk}{|k|^2}$$

This can be explicitly evaluated by using spherical coordinates for k , appropriately aligned. We will not compute this here.

Instead, we seek particular solution to (1.4) that reflects the $\frac{1}{|x|}$ scaling dependence manifest in (1.6) for velocity u . Because of high degree of symmetry, it is convenient to introduce spherical coordinates (r, θ, ϕ) for Navier-Stokes equation, where θ is the angle from the vector A , aligned along x_3 -axis. We look for solutions where spherical velocity components and pressure have the form:

$$(1.7) \quad (u_r, u_\theta, u_\phi, p) = \left(\frac{U(\theta)}{r}, \frac{V(\theta)}{r}, 0, \frac{P(\theta)}{r^2} \right)$$

The steady Stokes equation in spherical co-ordinates with azimuthal symmetry (*i.e.* no ϕ dependence) is given by

$$(1.8) \quad 0 = -\frac{\partial p}{\partial r} + \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r} \frac{\partial u_r}{\partial r} - \frac{2u_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{2}{r^2} \cot \theta u_\theta \right)$$

$$(1.9) \quad 0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2 \sin^2 \theta} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right)$$

$$(1.10) \quad \frac{\partial u_r}{\partial r} + \frac{2}{r} u_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\cot \theta}{r} u_\theta = 0$$

Using the form (1.7), we obtain

$$(1.11) \quad 0 = 2P + \{-2U + U'' + \cot \theta U' - 2V' - 2V \cot \theta\}$$

$$(1.12) \quad 0 = -P' + \left\{ -\frac{V}{\sin^2 \theta} + V'' + \cot \theta V' + 2U' \right\}$$

$$(1.13) \quad U + V' + V \cot \theta = 0$$

Eliminating U and P in (1.12) by using (1.11) and (1.12), we find:

$$(1.14) \quad \mathcal{L}_\theta V = 0$$

where the linear operator \mathcal{L}_θ is defined to be

$$(1.15) \quad \mathcal{L}_\theta V \equiv V^{(iv)} + 2 \cot \theta V''' - [2 + 3 \cot^2 \theta] V'' + [5 \cot \theta + 3 \cot^3 \theta] V' - 3(1 + \cot^2 \theta)^2 V$$

It is to be noted that four linearly independent solutions to $\mathcal{L}_\theta V = 0$ are given by

$$(1.16) \quad \sin \theta, \cot \theta, \frac{1 + \cos^2 \theta}{\sin \theta}, \log \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) \sin \theta$$

The only acceptable solution (without singularities in θ or multi-valuedness) is a multiple of $\sin \theta$. Therefore,

$$(1.17) \quad u_\theta(r, \theta) = \frac{C_1}{r} \sin \theta$$

Using (1.13) the radial velocity component

$$(1.18) \quad u_r(r, \theta) = -\frac{2C_1}{r} \cos \theta$$

We can determine P from (1.11) and obtain pressure

$$(1.19) \quad p(r, \theta) = -\frac{2C_1}{r^2} \cos \theta$$

Using the transformation between spherical and cartesian coordinates, it follows that

$$(1.20) \quad (u_1, u_2, u_3) = \{(u_\theta \cos \theta + u_r \sin \theta) \cos \phi, (u_\theta \cos \theta + u_r \sin \theta) \sin \phi, (-v_\theta \sin \theta + v_r \cos \theta)\}$$

It follows from the relation between spherical and cartesian coordinates that

$$(1.21) \quad u_1 = -\frac{C_1 x_3 x_1}{|x|^3}, \quad u_2 = -\frac{C_1 x_3 x_2}{|x|^3}, \quad u_3 = -\frac{C_1(x_3^2 + |x|^2)}{|x|^3}$$

and the pressure is given by

$$(1.22) \quad p = -\frac{2C_1 x_3}{|x|^3}$$

Noting that \mathbf{A} is aligned along x_3 axis, we may write the velocity u given in (1.21) in a coordinate free manner:

$$(1.23) \quad u = -C_1 \left(I + \frac{x(x \cdot)}{|x|^2} \right) \frac{A}{|A||x|}$$

while

$$(1.24) \quad p = -\frac{2C_1(A \cdot x)}{|x|^3|A|}$$

To determine C_1 , we go back to (1.4) and integrate the equation over a small sphere of radius ϵ ; this gives rise to the relation for the j component of A_j

$$A_j = \int_{|x|=\epsilon} \{pn_j - n_k \partial_{x_k} \mathbf{u}\} dx$$

Using (1.23) and (1.24) gives rise to to $C_1 = -\frac{|A|}{8\pi}$. So,

$$(1.25) \quad u = \frac{1}{8\pi|x|} \left(I + \frac{x(x \cdot)}{|x|^2} \right) \cdot A = G^S \cdot A$$

G^S is a tensor of rank 2, and is called the Stokeslet. Physically, G_{ij}^S is the i -component of velocity due to a delta function force at the origin oriented along the x_j -axis. More generally, if singular forcing is at $x = y$ instead of the origin, then

$$(1.26) \quad u(x) = \frac{1}{8\pi|x-y|} \left(I + \frac{(x-y)((x-y) \cdot)}{|x-y|^2} \right) \cdot A = G^S(x-y) \cdot A$$

For arbitrary force $b(x)$, we have

$$(1.27) \quad u(x) = \int_{y \in \mathbb{R}^3} G^S(x-y) \cdot b(y) dy,$$

where it is implicitly assumed that b decays fast enough at ∞ for the integral in (1.27) to exist. Notice the slow $1/r$ decay rate of a Stokeslet solution.

1.1.1. *Other Singular Solutions.* Since Stokes equation is linear, other singular solutions are useful as a linear combination is also useful. Instead of a delta-function force, we may consider the force dipole of strength \mathbf{p} at $x = y$. This corresponds to the solution when force is given by $b(x) = (p \nabla_y) \delta(x - y)$. This singular solution, is easily obtained from Stokeslet solution

$$(1.28) \quad u(x) = p \cdot [\nabla_y G^S(x - y)] \cdot A$$

The third-order tensor

$$(1.29) \quad G^{sd}(x - y) \equiv \nabla_y G^S(x - y)$$

is called a *Stokeslet doublet*. Similarly, more singular Stokeslet multiplet solutions can be found by applying ∇_y . It is to be noted that, while these are more singular at $x = y$, they decay faster at ∞ .

Another physically important solution is the point Source and dipole solutions for $b = 0$. It is to be noted from applying the curl operator on (1.3) that any potential flow solution, *i.e.* $u = \nabla \phi$ is a solution to (1.3). For potential flow, a flow of particular importance is the source solution. We saw earlier that in 2-D, this source solution of strength m at the origin is given by $u = \nabla \frac{m}{2\pi} \log |x|$. Not surprisingly, the source solution in 3-D is given by

$$(1.30) \quad u(x) = \nabla \frac{-m}{4\pi|x|} = \frac{mx}{4\pi|x|^3} = mG^p(x)$$

To show that this corresponds to fluid flowing out from the origin at rate m we calculate

$$(1.31) \quad \int_{|x|=\epsilon} (u \cdot n) dx = 4\pi\epsilon^2 \frac{m}{4\pi\epsilon^2} = m$$

More generally, $G_p(x - y)$ is the point source solution, corresponding to a physical source spewing out fluid at $x = y$ at unit rate.

Another singular solution that is physically relevant is the source dipole solution

$$(1.32) \quad G^d(x - y) \equiv \nabla_y G^S(x - y) = \frac{1}{4\pi|x - y|^3} \left(-I + 3 \frac{(x - y)([x - y] \cdot \cdot)}{|x - y|^2} \right)$$

The velocity generated by a point dipole at y of strength p is given by

$$(1.33) \quad u(x) = [G^d(x - y)] \cdot p$$

The dipole can be thought of a source and sink of equal and opposite strength brought together so that in the limit of small separation, the line joining the sink to the source is oriented in the direction of p , and the product of source strength and source-sink-separation is $|p|$.

1.2. Steady Stokes Flow past a Sphere. By taking an appropriate linear combination of singular solution to Stokes equation, just discussed, we can arrive at the problem of finding flow past a moving sphere of radius a , moving with velocity U_0 . Indeed, we can check directly that

$$(1.34) \quad u(x) = \left[G^S(x) - \frac{a^2}{6} G^d(x) \right] \cdot A$$

satisfies the equation and the boundary condition on $|x| = a$:

$$(1.35) \quad u = U_0$$

when

$$(1.36) \quad A = 6\pi a U_0$$

In dimensional form, this says that the drag force on the sphere is $6\pi a \mu U_0$, a result that is sometimes taught in high school.

It is to be noted that in this case, there is no singularity in the flow region $|x| \geq a$. Superposition of singular solutions at $x = 0$ allowed us to solve this problem. A direct method of solving this problem will be to use spherical coordinates in the domain $r > a$, in the same manner as we obtained the Stokeslet solution G^S .

Also, since $G^S = O(\frac{1}{r})$, while $G^d = O(\frac{1}{r^3})$, for large distances r from the origin, the flow is essentially the same as for a Stokeslet. Bodies of more complicated shapes give rise to higher order multipoles; nonetheless, as $x \rightarrow \infty$, the flow approaches once again a Stokeslet. This observation has important consequences when we are trying to model the flow past many spheres. We can think of the presence of far-away spheres as Stokeslet of appropriate strength. This makes computation manageable.

1.3. Solid Wall effect on Stokeslet Flow. Consider a Stokeslet in the presence of a wall that is either parallel or perpendicular to the force A . In this case, we have a Stokeslet at y in the domain

$$\Omega = \{x \in \mathbb{R}^3, x_3 > 0\}$$

We need to satisfy the no-slip condition at $x_3 = 0$. We can check directly that the solution in this case is

$$(1.37) \quad u = [G^S(x - y) + G^W(x - y')] \cdot A, \text{ where } y = (y_1, y_2, y_3), \ y' = (y_1, y_2, -y_3);$$

and

$$(1.38) \quad G^W(x - x') = -G^S(x - y') \pm [y_3^2 G^d(x - y') - (0, 0, 2y_3) \cdot G^{SD}(x - y')]$$

The plus sign is if the Stokeslet is oriented parallel to the wall, and negative if it is oriented perpendicular to the wall.