Week 8 lectures

1. Equations for motion of fluid without incompressible Assumptions

Recall from week 1 notes, the equations for conservation of mass and momentum, derived generally without any incompressibility assumption leads to

(1.1)
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

(1.2)
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{b},$$

where \mathbf{T} is the second order Stress Tensor. For *Newtonian* fluids,

(1.3)
$$\mathbf{T} = 2\mu \left(\mathbf{S} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \mathbf{I} \right) + \left[\lambda + \frac{2}{3} \mu \right] (\nabla \cdot \mathbf{u}) \mathbf{I},$$

where μ and λ are viscosity and second viscosity coefficients, respectively and **S** is the strain tensor:

(1.4)
$$S_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \equiv \frac{1}{2} \left(\partial_{x_j} u_i + \partial_{x_i} u_j \right)$$

Using (1.3) and (1.4), (1.2) becomes

(1.5)
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \rho \mathbf{b}$$

In general, (1.1) and (1.5) are not enough to determine the flow since these constitute four scalar equations for five scalar unknowns \mathbf{u}, ρ, p . Additional equations are needed, which come from energy conservation and thermo-dynamics as in the ensuing.

2. Energetic Considerations

Consider an arbitrary but fixed volume $W \subset \Omega \subset \mathbb{R}^3$ with smooth boundary having outwards normal **n**. Consider rate of change of energy contained in W:

(2.6)
$$\frac{d}{dt} \int_{W} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho \mathcal{E} \right) dx = \int_{W} \partial_t \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho \mathcal{E} \right), dx$$

where $\mathcal{E}(\mathbf{x}, t)$ is the internal energy perunit mass at a location \mathbf{x} at time t. The inwards flux of energy through ∂W , which includes heat

flux as well⁽¹⁾, is given by

(2.7)
$$-\int_{\partial W} \left(\frac{1}{2}\rho |\mathbf{u}|^2 + \rho \mathcal{E}\right) \mathbf{u} \cdot \mathbf{n} dx + \int_{\partial W} \mathbf{n} \cdot (k\nabla T) dx$$
$$= -\int_{W} \left\{ \nabla \cdot \left(\mathbf{u} \left[\frac{1}{2}\rho |\mathbf{u}|^2 + \rho \mathcal{E}\right] \right) - \nabla \cdot (k\nabla T) \right\} dx,$$

where $T(\mathbf{x}, t)$ is the temperature at position \mathbf{x} at time t and k is the thermal conductivity. Now consider the work done by forces in the system. It is convenient to define the total Stress tensor Σ so that

(2.8)
$$\Sigma_{ij} = -p\delta_{i,j} + T_{ij}$$

The work done on the fluid inside W in unit time, including body and surface forces, is given by

(2.9)
$$\int_{W} \rho \mathbf{u} \cdot \mathbf{b} dx + \int_{\partial W} u_i \Sigma_{ij} n_j dx = \int_{W} \left(\rho u_i b_i + u_{i,j} \Sigma_{ij} + u_i \Sigma_{ij,j} \right) dx$$

From conservation of energy, the sum of rate of change of energy given by (2.6) equals the sum of inwards energy flux (2.7) and the work done by forces per unit time given by (2.9). This is true for arbitrary W and so

(2.10)
$$\partial_t \left(\frac{1}{2}\rho |\mathbf{u}|^2 + \rho \mathcal{E}\right) + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{1}{2}\rho |\mathbf{u}|^2 + \rho \mathcal{E}\right)\right] - \nabla \cdot (k\nabla T) = u_i b_i + u_{i,j} \Sigma_{ij} + u_i \Sigma_{ij,j}$$

Using (1.1) and (1.2), noting that $\Sigma_{ij} = -pn_j + T_{ij}$, we obtain

(2.11)
$$\frac{D\mathcal{E}}{Dt} = \frac{1}{\rho} u_{i,j} \Sigma_{ij} + \frac{1}{\rho} \nabla \cdot (k \nabla T),$$

where we recall operator $\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla$.

3. Thermodynamic consideration

We now recall some facts from equilibrium Thermodynamics⁽²⁾. The intensive quantities used include internal energy per unit mass \mathcal{E} , temperature T, pressure p, specific volume $v = \frac{1}{\rho}$, *i.e.* volume of fluid per unit mass, entropy per unit mass S, which is a characteristic of disorder in the system⁽³⁾ The fundamental ansatz in thermodynamics, which is supported by latter theoretical developments in statistical mechanics as

⁽¹⁾This is because according to first law of thermodynamics, heat inputted into the system is transferred to work done by the system and change of internal energy.

⁽²⁾ For most purposes it is good enough to ignore non-equilibrium effects since the thermodynamic time-scale is far shorter than the time scale in which fluid moves

⁽³⁾Using statistical mechanics, entropy can be quantified to be proportional to the log of the number of states in the system consistent with a given thermodynamic state.

well as experiment, is that any two of these variables completely characterize a system. For instance, if T and v are considered as independent variables, the rest of the variables p, S, \mathcal{E} , etc., are each functions of Tand v. Choice of different independent variables is suitable for different purposes.

The first law of thermodynamics states that at each location occupied by a gas or fluid,

$$(3.12) d\mathcal{E} = TdS - pdv,$$

where dq = TdS denotes the infinitesimal heat generated, dw = -pdvis the infinitesimal work done on the system, per unit time. What is important about the relation (3.12) is that \mathcal{E} is a state variable, which mathematically means that $d\mathcal{E}$ is an exact differential, where in (3.12) we have used S and v as the two independent variables. Therefore, it follows from (3.12) that $\frac{\partial \mathcal{E}}{\partial S} = T$ and $\frac{\partial \mathcal{E}}{\partial v} = -p$. From equality of mixed second partials, where we assume \mathcal{E} to be smooth enough function of S and v, it follows that

(3.13)
$$\left(\frac{\partial T}{\partial v}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_v$$

 \mathcal{E} is not the only state variable. $I = \mathcal{E} + pv$ called *enthalpy* per unit mass is another state variable. If we consider instead S and p to be two independent variables, it follows from (3.12) that

(3.14)
$$dI = d\left(\mathcal{E} + pv\right) = TdS + vdp$$

Equality of two second mixed partials of I gives

(3.15)
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial v}{\partial S}\right)_p$$

Yet another state variable is the *Gibbs free energy* $F = \mathcal{E} - TS$ per unit mass. If we use v and T as two independent variables, then (3.12) implies

$$(3.16) dF = -pdv - SdT$$

The equality of mixed second partials of F gives

(3.17)
$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial S}{\partial v}\right)_T$$

Again, F + pv is also a state variable. If we use p and T as independent variable, it follows from (3.16) that

$$(3.18) d(F+pv) = vdp - SdT,$$

equality of mixed partials immediately implies

(3.19)
$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$$

It is also convenient to define

(3.20)
$$c_p = T\left(\frac{\partial S}{\partial T}\right)_p,$$

(3.21)
$$c_v = T \left(\frac{\partial S}{\partial T}\right)_v$$

which is the *specific heat* per unit mass for constant pressure and constant volume respectively, since dQ = TdS is the heat content⁽⁴⁾ per unit mass per unit temperature. We also define *volumetric expansion* rate with temperature for fixed pressure

(3.22)
$$\beta = \left(\frac{\partial v}{\partial T}\right)_p$$

4. Full equation of fluid dynamics

We return to (2.11). We note from (3.12) that

(4.23)
$$T\left(\frac{\partial S}{\partial T}\right)_{p}\frac{DT}{Dt} + T\left(\frac{\partial S}{\partial p}\right)_{T}\frac{Dp}{Dt} = T\frac{DS}{Dt} = \frac{D\mathcal{E}}{Dt} + p\frac{D}{Dt}\frac{1}{\rho}$$

Using (3.19), (3.20) and (4.23) and (2.11), we obtain

(4.24)
$$c_p \frac{DT}{Dt} - \beta T \frac{Dp}{Dt} = T \frac{DS}{Dt} = \frac{1}{\rho} u_{i,j} \Sigma_{ij} + \frac{1}{\rho} \nabla \cdot (k \nabla T) + \frac{p}{\rho} \nabla \cdot \mathbf{u}$$

Using (1.3) and (2.8), it follows that

(4.25)
$$T\frac{DS}{Dt} = c_p \frac{DT}{Dt} - \beta T \frac{Dp}{Dt} = \Phi + \frac{\lambda + \frac{2}{3}\mu}{\rho} \left(\nabla \cdot \mathbf{u}\right)^2 + \nabla \cdot \left(k\nabla T\right),$$

where the positive definite quantity (4.26)

$$\Phi = \frac{2\mu}{\rho} \left(S_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \delta_{i,j} \right) \left(S_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \delta_{i,j} \right) = \frac{2\mu}{\rho} \left(S_{i,j} S_{i,j} - \frac{1}{3} (\nabla \cdot \mathbf{u})^2 \right)$$

has the physical interpretation of dissipation of mechanical energy perunit mass As stated earlier, we also have an equation of state that follows from equilibrium thermodynamics

(4.27)
$$f(p, \rho, T) = 0$$

⁽⁴⁾ Note that in our formulation the unit of heat and unit of energy are chosen the same

Equations (1.1), (1.5), (4.25) and (4.27) constitute a complete set of six scalar equations for the unknowns ρ , p, T and \mathbf{u} , since the viscosity coefficients $\mu = \mu(\rho, T)$, $\lambda = \lambda(\rho, T)$ and thermal conductivity $k = k(\rho, T)$ are known functions.

4.1. Bernoulli law for steady flow of a frictionless non-conducting fluid. Assume that the force perunit mass $\mathbf{b} = -\nabla \Psi$, is time-independent. Then, from (2.10), using (1.1),

(4.28)
$$\frac{D}{Dt}\left(\mathcal{E} + \frac{1}{2}|\mathbf{u}|^2\right) = -\mathbf{u}\cdot\nabla\Psi + \frac{1}{\rho}\partial_{x_j}\left(u_i\Sigma_{ij}\right) + \frac{1}{\rho}\nabla\cdot\left(k\nabla T\right)$$

Using (1.3), (1.4) and (1.1), it follows that (4.29)

$$\frac{1}{\rho}\partial_{x_{j}}\left(u_{i}\Sigma_{ij}\right) = -\frac{1}{\rho}\mathbf{u}\cdot\nabla p - \frac{p}{\rho}\nabla\cdot\mathbf{u} + \frac{2\mu}{\rho}\partial_{x_{j}}\left\{u_{i}\left[S_{ij} - \frac{1}{3}\left(\nabla\cdot\mathbf{u}\right)\delta_{i,j}\right]\right\} + \left(\lambda + \frac{2}{3}\mu\right)\left(\nabla\cdot\mathbf{u}\right)^{2}$$
$$= -\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{2\mu}{\rho}\partial_{x_{j}}\left\{u_{i}\left[S_{ij} - \frac{1}{3}\left(\nabla\cdot\mathbf{u}\right)\delta_{i,j}\right]\right\} + \left(\lambda + \frac{2}{3}\mu\right)\partial_{x_{j}}\left(u_{j}\nabla\cdot\mathbf{u}\right)$$

Therefore, for a frictionless non-conducting fluid limit, *i.e.* when viscosity coefficients μ , λ , as well as conductivity coefficient k is negligible, and the flow is steady, *i.e.* $p_t = 0$, we have from (4.28) and (4.29),

(4.30)
$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2 + \mathcal{E} + \frac{p}{\rho} + \Psi\right) = 0$$

This means that along a streamline:

(4.31)
$$H = \frac{1}{2} |\mathbf{u}|^2 + \mathcal{E} + \frac{p}{\rho} + \Psi$$

is a constant; this is generalization of Bernoulli's principle for a general compressible fluid that is frictionless, non-conducting and steady.

5. ISENTROPIC AND HOMENTROPIC FLOWS

We note from (4.25) if the positive definite dissipation terms Φ , $\left(\lambda + \frac{2}{3}\mu\right)(\nabla \cdot \mathbf{u})^2$ are small, along with temperature diffusion term $\nabla \cdot (k\nabla T)$, each of which is the result of molecular diffusion, then we can approximate $\frac{DS}{Dt} = 0$, *i.e. entropy* is constant along a stream line, though it need not be the same constant on different straight lines. When the latter is true, the flow is said to be *homentropic*. For isentropic flow, it is useful to take equation of state in the form

$$(5.32) \qquad \qquad \rho = \rho(p, S)$$

and define

(5.33)
$$\left(\frac{\partial p}{\partial \rho}\right)_S = c^2$$

Then ignoring molecular diffusion terms in (1.5), which is consistent with isentropic approximation process, (1.1) and (1.5) simplify under isentropic approximation to the follow relation

(5.34)
$$\frac{1}{\rho c^2} \frac{Dp}{Dt} + \nabla \cdot \mathbf{u} = 0$$

(5.35)
$$\frac{D\mathbf{u}}{Dt} = \mathbf{b} - \frac{1}{\rho}\nabla p$$

Equations (5.34), (5.35) along with $\frac{DS}{Dt} = 0$ and equation of state (4.27) determine **u**, *p*, *S* and ρ for an *isentropic* flow.

For a homentropic flow, since S is a constant everywhere, $p = p(\rho)$ and $c^2 = c^2(\rho(p))$, in which case the set of equations (5.34) and (5.35) completely determine the unknowns p and **u**; or equivalently ρ and **u** if we choose to rewrite (5.34) and (5.35) in the form

(5.36)
$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

(5.37)
$$\frac{D\mathbf{u}}{Dt} = \mathbf{b} - \nabla w \text{, where } w = \int^{\rho} \frac{c^2(\rho')}{\rho'} d\rho'$$

As shall be seen later, these form a pair of nonlinear hyperbolic PDEs; the solutions are typically characterized by *shocks*, *i.e.* solutions need to be weak solutions with jumps, and classical solutions have to be matched across the two sides of the shocks by considering conserved quantities on stream lines.

Note from (4.23),

(5.38)
$$\frac{D}{Dt}\left(\mathcal{E} + \frac{p}{\rho}\right) = T\frac{DS}{Dt} + \frac{1}{\rho}\frac{Dp}{dt}$$

Therefore, it follows that for Isentropic flows, the Bernoulli principle for steady flow (4.30) reduces to

(5.39)
$$H = \frac{1}{2} |\mathbf{u}|^2 + \int^{\rho} \frac{c^2(\rho', S)}{\rho'} d\rho' + \Psi = \text{constant on astreamline}$$

Note that if a flow has a shock region, molecular effects like viscous dissipation and heat conduction become important; so typically H will jump across the shock by an amount depending on how much dissipation and heat conduction takes place in a thin shock region.

5.1. Sound propagation in homentropic flows. Consider small departure from an equilibrium flow where $\mathbf{u} = 0$. In equilibrium (5.37) implies $p = p_0$

(5.40)
$$\mathbf{b} = \frac{1}{\rho_0} \nabla p_0 = \nabla w_0$$

Such an equilibrum is when ${\bf b}$ is time-independent and curl-free. Then, if we decompose

(5.41)
$$p = p_0 + p_1, \rho = \rho_0 + \rho_1$$

and linearize for small p_1 , ρ_1 and \mathbf{u} , then (5.34) and (5.35) imply

(5.42)
$$\frac{1}{\rho_0 c_0^2} \partial_t p_1 + \nabla \cdot \mathbf{u} = 0$$

(5.43)
$$\rho_0 \mathbf{u}_t = \rho_1 \mathbf{b} - \nabla p_1$$

Dividing (5.43) by ρ_0 and taking divergence of the resulting equation, while taking the time derivative of (5.42), we obtain on elimination of **u**

(5.44)
$$\frac{1}{\rho_0 c_0^2} \frac{\partial^2 p_1}{\partial t^2} = \nabla \cdot \left(\frac{1}{\rho_0} \nabla p_1\right) - \nabla \cdot \left(\frac{\rho_1}{\rho_0} \mathbf{b}\right)$$

This is the equation for propagation of sound. If ρ_0 is spatially uniform, and effect of force **b** is negligible, we obtain the usual wave equation

(5.45)
$$\frac{1}{c_0^2} \frac{\partial^2 p_1}{\partial t^2} = \Delta p_1,$$

where c_0 will be the speed of propagation of sound, as we know from elementary PDE class. We note that sound propagation is possible in a medium with compressibility taken into account.

5.2. Vorticity in homentropic flow and potential flow approximation. We recall the identity

(5.46)
$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity, it follows from taking curl of (5.37), recalling $\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$,

(5.47)
$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nabla \times \mathbf{b}$$

If force **b** is conservative, we note that if $\boldsymbol{\omega} = 0$ initially, then $\boldsymbol{\omega} = 0$ for all $t^{(5)}$, *i.e.* a potential flow approximation $\mathbf{u} = \nabla \phi$ is valid under those conditions. In this case, for steady flow, Bernoulli law (show

 $^{^{(5)}}$ This assumes uniqueness of solutions, which is true, but is yet to be proved.

as an exercise) (5.39) is globally valid, *i.e* H is the same constant everywhere.

5.3. Validity of incompressible $\nabla \cdot \mathbf{u} = 0$ approximation. We return to the full set of equations (1.1),(1.5), (4.25) and (4.27) for ρ , p, T and \mathbf{u} and examine in a formal sense the conditions under which incompressibility $\nabla \cdot \mathbf{u} = 0$ assumptions would remain valid. We denote a typical velocity scale by scalar U and a typical length scale of variation of \mathbf{u} to be L. So, from (1.1), incompressibility assumption is valid if

(5.48)
$$\left|\frac{1}{\rho}\frac{D\rho}{Dt}\right| << \frac{U}{L}$$

If we choose ρ and *entropy* per unit mass S as independent variables, then

(5.49)
$$\frac{Dp}{Dt} = \left(\frac{\partial p}{\partial \rho}\right)_{S} \frac{D\rho}{Dt} + \left(\frac{\partial p}{\partial S}\right)_{\rho} \frac{DS}{Dt} = \frac{1}{c^{2}} \frac{D\rho}{Dt} + \left(\frac{\partial p}{\partial S}\right)_{\rho} \frac{DS}{Dt}$$

So condition (5.48) translates to

(5.50)
$$\left|\frac{1}{\rho c^2}\frac{Dp}{Dt} - \frac{1}{\rho c^2}\left(\frac{\partial p}{\partial S}\right)_{\rho}\frac{DS}{Dt}\right| <<\frac{U}{L}$$

This will be satisfied if each of the conditions $\left|\frac{1}{\rho c^2} \frac{Dp}{Dt}\right| << \frac{U}{L}$ and $\left|\frac{1}{\rho c^2} \left(\frac{\partial p}{\partial S}\right)_{\rho} \frac{DS}{Dt}\right| << \frac{U}{L}$. are satisfied. Normally, in estimating $\frac{Dp}{Dt}$, we may assume flow to be *isentropic* since of molecular diffusion effects like viscosities and thermal conductivity only effect distribution of pressure gradient rather than its order of magnitude. Then using (5.35), it follows that the condition on pressure becomes

(5.51)
$$\left|\frac{1}{\rho c^2}\frac{\partial p}{\partial t} - \frac{1}{2c^2}\frac{D}{Dt}|\mathbf{u}|^2 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{b}\right| <<\frac{U}{L}$$

Noting that time scale associated with choice of velocity scales U and L is $\frac{L}{U}$, the second term term $\left|\frac{1}{2c^2}\frac{D}{Dt}\|\mathbf{u}\|^2\right| << \frac{U}{L}$, when $U^2/c^2 << 1$.

Now conider the magnitude of the first term on the left of (5.51), which depends on the unsteadiness of the flow. If the flow-field is oscillatory with frequency n, noting that p/ρ scales as U^2 , then the condition $\frac{1}{\rho c^2} \frac{\partial p}{\partial t} \ll \frac{U}{L}$ will be satisfied when $\frac{n^2 L^2}{c^2} \ll 1$. If $n = O(\frac{U}{L})$, then the above condition is satisfied when $U^2/c^2 \ll 1$. If $n \gg \frac{U}{L}$, then the condition is more exacting.

Consider the last term in (5.51). It is clear that $\frac{\mathbf{u} \cdot \mathbf{b}}{c^2} \ll \frac{U}{L}$ if $\frac{|\mathbf{b}|L}{c^2} \ll 1$. If **b** is gravity, this condition is satisfied in most laboratory conditions, given the relatively large value of speed of sound.

We now consider the second term on the left of (5.50) with isentropic assumption. We first note that from thermodynamic relations derived earlier

(5.52)
$$\frac{1}{\rho c^2} \left(\frac{\partial p}{\partial S}\right)_{\rho} = -\frac{1}{\rho c^2} \left(\frac{\partial p}{\partial \rho}\right)_S \left(\frac{\partial \rho}{\partial S}\right)_p = -\frac{\left(\frac{\partial \rho}{\partial T}\right)_p}{\rho \left(\frac{\partial S}{\partial T}\right)_p} = \frac{\beta T}{c_p}$$

Using (4.25), it follows from (5.52) that the second term on the left of (5.50) $<<\frac{U}{L}$ if

(5.53)
$$\left|\frac{\beta}{c_p}\left\{\Phi + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)\right\}\right| << \frac{U}{L},$$

which is physically the statement that variation of density of a material element in the fluid due to internal dissipation and conduction of heat is small.