Week 9 lectures

1. Isentropic 1-D Gas Dynamics Equations

In one space dimension, the equations for isentropic fluid dynamics (5.34)-(5.35) of week 8 notes become

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{1.1}$$

$$(u_t + uu_x) + \frac{1}{\rho} p_x = 0 \tag{1.2}$$

$$S_t + uS_x = 0, \tag{1.3}$$

where from equilibrium thermodyanmics, $p = p(\rho, S)$ is considered known. Since $dp = \left(\frac{\partial p}{\partial S}\right)_\rho dS + c^2(\rho, S) d\rho$, it follows from (1.3) and (1.1) that

$$p_t + up_x = c^2 \left(\rho_t + u\rho_x\right) = -c^2 \rho u_x \tag{1.4}$$

Between (1.2) and (1.4), we seek to take linear combination of the two equations to find suitable characteristic variables. If we multiply (1.2) by ρc and add/subtract from (1.4), then we obtain

$$(p_t + (u+c)p_x) + \rho c \left(u_t + (u+c)u_x\right) = 0 \tag{1.5}$$

$$(p_t + (u-c)p_x) - \rho c \left(u_t + (u-c)u_x\right) = 0 \tag{1.6}$$

Equations (1.5)-(1.6) in addition to (1.3) are the complete set of hyperbolic equations for determination of p, u and S is characteristic form since we may write them as following set of ODEs on characterics

$$\frac{dp}{dt} + \rho c \frac{du}{dt} = 0 \ , \ \text{on} \ \frac{dx}{dt} = u + c \tag{1.7}$$

$$\frac{dp}{dt} - \rho c \frac{du}{dt} = 0 \ , \ \text{on} \ \frac{dx}{dt} = u - c \tag{1.8}$$

$$\frac{dS}{dt} = 0 \ , \text{on} \ \frac{dx}{dt} = u \tag{1.9}$$

Note the characteristic speeds are u, $u \pm c$. The latter two are associated with sound waves when fluid motion is present. If we linearize the nonlinear equations (1.7)-(1.9) about a quiescent state $u = 0$ with uniform density ρ_0, we obtain

$$\frac{dp}{dt} + \rho_0 c_0 \frac{du}{dt} \ \text{on} \ \frac{dx}{dt} = c_0 \tag{1.10}$$

$$\frac{dp}{dt} - \rho_0 c_0 \frac{du}{dt} \ \text{on} \ \frac{dx}{dt} = -c_0 \tag{1.11}$$

$$\frac{dS}{dt} = 0 \text{ on } \frac{dx}{dt} = 0 \tag{1.12}$$

Integrating (1.12),

$$S - S_0 = H(x) \tag{1.13}$$

In the case of a *homentropic* flow, $H(x) = 0$. In that case, integration of (1.10) and (1.11) results in

$$p - p_0 + \rho_0 c_0 u = F(x - c_0 t) \ , \tag{1.14}$$

$$p - p_0 - \rho_0 c_0 u = G(x + c_0 t) \ , \tag{1.15}$$

implying

$$p - p_0 = \frac{1}{2}\left(F(x - c_0 t) + G(x + c_0 t)\right) \tag{1.16}$$

$$u = \frac{1}{2\rho_0 u_0}\left(F(x - c_0 t) + G(x + c_0 t)\right) \tag{1.17}$$

Returning to nonlinear equations (1.7)-(1.9), if we have a *homentropic* flow, then $p = p(\rho)$, $c^2 = p'(\rho)$ and it follows that

$$\frac{d}{dt}\left(\int^{\rho} \frac{c(\rho')}{\rho'} d\rho' + u\right) = 0 \text{ , on } \frac{dx}{dt} = u + c \tag{1.18}$$

$$\frac{d}{dt}\left(\int^{\rho} \frac{c(\rho')}{\rho'} d\rho' - u\right) = 0 \text{ , on } \frac{dx}{dt} = u - c \tag{1.19}$$

Recall for that for a gas $p = C\rho^{\gamma}$ and so, $c^2 = \gamma C \rho^{\gamma - 1}$; using this (1.18) and (1.19) implies

$$\frac{d}{dt}\left(\frac{2c}{\gamma - 1} + u\right) = 0 \text{ , on } \frac{dx}{dt} = u + c \tag{1.20}$$

$$\frac{d}{dt}\left(\frac{2c}{\gamma - 1} - u\right) = 0 \text{ , on } \frac{dx}{dt} = u - c \tag{1.21}$$

and we have on integration the Riemann invariants

$$\frac{2}{\gamma - 1} c \pm u = \text{constant} \ , \text{ on } \frac{dx}{dt} = u \pm c \tag{1.22}$$

2. Case of a moving piston

The equations (1.22) simplify further if initial conditions are such that one of the Riemann invariants is trivial, as it can be used in the other to turn it into a first order nonlinear PDE for one scalar variable. This is the case for the piston problem as described below. When such simplifications are not possible, the Riemann invariants (1.22) can still be used to compute or analyze solutions.

Consider an initially quiescent gas $u = 0$ in an semi-infinite cylindrical container where the piston boundary is at $x = X(t)$, with $X(0) = 0$. We assume initial density and entropy to be $\rho = \rho_0$, $S = S_0$, each of which are constants. The domain in $t - x$ plane is shown in Figure 1.

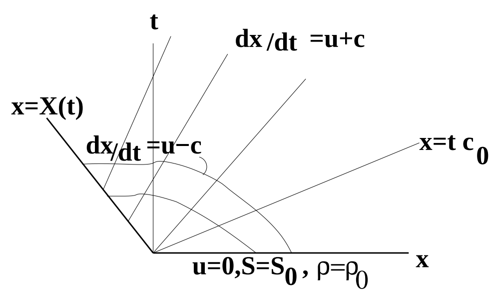

Figure 1. Two sets of characteristics shown in the $t - x$ domain to the right of a moving piston $x = X(t)$

On the set of characteristics $\frac{dx}{dt} = u - c$, which intersects the piston $x = X(t)$ as well as the x-axis, where $u = 0$, $c = c_0$,

$$\frac{2c}{\gamma - 1} - u = \frac{2c_0}{\gamma - 1} \tag{2.23}$$

This can be used to eliminate c completely from the equations. Therefore, on the characteristic set

$$\frac{dx}{dt} = u + c = c_0 + \frac{\gamma + 1}{2} u, \tag{2.24}$$

we obtain from the Riemann invariant

$$\frac{2c}{\gamma - 1} + u = 2u + \frac{2c_0}{\gamma - 1} = \text{constant} \tag{2.25}$$

or equivalently

$$u_t + \left(c_0 + \frac{\gamma + 1}{2} u \right) u_x = 0 \tag{2.26}$$

In the region in the $t-x$ plane region, where characteristic set (2.24) intersects $x = X(t)$, (2.25) implies

$$u = u\Big|_{\text{piston}} = \dot{X}(\tau) \tag{2.27}$$

where $t = \tau$ is the initial value of t where the set of characteristics (2.24) intersects intersects $x = X(t)$. Using (2.27), integration of (2.24) results in

$$x = X(\tau) + \left(c_0 + \frac{\gamma+1}{2}\dot{X}(\tau) \right)(t-\tau) \tag{2.28}$$

If and when it is possible to invert (2.28) to solve $\tau = \tau(x,t)$ it produces a classical solution to the PDE (2.26) in the form

$$u = \dot{X}\left(\tau(x,t)\right) \ , \tag{2.29}$$

By taking partial derivative of (2.28) with respect to τ, the implicit function theorem condition of inversion is:

$$-\frac{\gamma-1}{2}\dot{X}(\tau) - c_0 + \frac{(\gamma+1)}{2}\ddot{X}(\tau)\,(t-\tau) \neq 0 \tag{2.30}$$

For a piston moving to the left, *i.e.* $\dot{X} < 0$, we still have $-\frac{\gamma-1}{2}\dot{X}(\tau) - c_0 < 0$ when the piston speed is not comparable to sound speed. Under these conditions, if $\ddot{X} \leq 0$, then (2.30) is satisfied since each term is < 0. In the special case when $\dot{X} = -V =$ constant, then solution in the region $x < \left(c_0 - \frac{\gamma+1}{2}V\right)t$ is simply

$$u(x,t) = -V_0 \ , c = c_0 - \frac{(\gamma-1)}{2}V \tag{2.31}$$

and for $x > c_0 t$,

$$u(x,t) = 0 \ , c(x,t) = c_0 \tag{2.32}$$

In the intermediate region $c_0 t \geq x \geq \left(c_0 - \frac{\gamma+1}{2}V\right)t$, we have a *simple wave* solution

$$u(x,t) = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right) \tag{2.33}$$

which may be verified to be a solution of the PDE (2.26) for $t > 0$ by direct substitution. These are continuous solutions. However, if $\ddot{X}(\tau) > 0$ for any τ, then it is easily seen inversion condition (2.30) will not always be valid. This gives rise to characteristic curves (2.28) intersecting, which corresponds to classical solution developing singularities. Beyond such time, physically reasonable solutions are ones where one allows solutions to undergo jumps, *i.e.* we allow for weak solutions.

2.1. **Shock Waves.** When waves break, *i.e.* singularities form in classical solution, then the inversion condition (2.30) becoming invalid. In that case, we must return to the basic derivation of conservation of mass momentum and energy for $x \in (x_1, x_2)$ to determine additional conditions that determine where to place one or more shocks *i.e.* discontinuities across which ρ, u and S may jump. We study these relations in one space dimension for simplicity, though the idea is much more general. Conservation of mass from week 1 notes implies

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx + [\rho u]_{x_1}^{x_2} = 0 \ , \tag{2.34}$$

Conservation of momentum in the absence of body forces implies

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u dx + \left[\rho u^2 + p\right]_{x_1}^{x_2} = \left(\frac{2}{3}\mu + \lambda\right)\left[\frac{\partial u}{\partial x}\right]_{x_1}^{x_2} \ , \tag{2.35}$$

Conservation of energy implies

(2.36)

$$\frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}\rho u^2 + \rho \mathcal{E}\right)dx + \left[\left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right)u + pu\right]_{x_1}^{x_2} = \left[k\frac{dT}{dx} + \left(\frac{2}{3}\mu + \lambda\right)\frac{du}{dx}\right]_{x_1}^{x_2}$$

where $[.]_{x_1}^{x_2}$ is the evaluation of the quantity at x_1 subtracted from evaluation at at x_2. The terms on the right in (2.35) and (2.36) are due to molecular diffusion effects and can be ignored outside of a shock region. If $x = X_s(t)$ denotes the location of a shock where there is discontinuity of flow quantities, then by taking $x_2 = X_s(t) + \epsilon$ and $x_1 = X_s(t) - \epsilon$ and taking the limit of $\epsilon \to 0^+$, with $U = \dot{X}_s$, while ignoring molecular effects, it follows that

$$-U[\rho] + [\rho u] = 0 \ , \tag{2.37}$$

$$-U[\rho u] + \left[\rho u^2 + p\right] = 0 \ , \tag{2.38}$$

$$-U\left[\frac{1}{2}\rho u^2 + \rho c\right] + \left[\left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right)u + pu\right] = 0 \ , \tag{2.39}$$

One avoids the problem of overlapping characteristics for the case $\ddot{X} > 0$ by enforcing conditions (2.37)-(2.39) across the two sides of a shock $x = X(\tau)$; away from the shock one uses the classical solution such as the one obtained in (2.29). In this context, it is useful to note that for a *polytropic* gas under conditions of isentropic flow, $\mathcal{E} = \frac{p}{(\gamma-1)\rho}$. We avoid going through any more details here, though there are excellent texts on the subject of shocks (see for instance Whitham, Linear and Nonlinear Waves, Wiley).

3. Incompressible free boundary problems: Motion of Bubbles

Thus far, we have discussed incompressible flow in a fixed domain Ω. There are many instances, when this is not appropriate. For instance in the motion of a bubble, the boundary itself evolves in time. The same is true for water waves. On a free boundary, such as between two fluids, or between fluid and vacuum, the no-slip condition $\mathbf{u} = \mathbf{v}$, used for a solid boundary, is no longer appropriate. Consider first the case of a fluid boundary $\partial\Omega$ with another fluid of negligible viscosity and pressure variation, *i.e.* vacuum conditions. Since an infinitesimal free boundary has infinitesimal mass, the forces on the two sides must be in balance. This corresponds to[(1)]

$$-pn_i + 2\mu T_{ij} n_j = -pn_i + \mu \left(\partial_{x_j} u_i + \partial_{x_i} u_j \right) n_j = -\sigma \kappa n_i \tag{3.40}$$

where σ is the surface tension and κ the curvature for in 2-D, and mean curvature in 3-D. Note $\mathbf{T}$ is the viscous stress tensor and (3.40) is a vector relation. It is called the *Stress condition.* By taking the dot product of relation (3.40) with $\mathbf{n}$, we obtain

$$-p + n_i T_{i,j} n_j = -\sigma\kappa \text{ ,on} \partial\Omega \tag{3.41}$$

By taking dot product with respect to a tangent vector τ, tangent to the interface for which $\boldsymbol{\tau} \cdot \mathbf{n} = 0$, we obtain from (3.40)

$$\tau_i T_{i,j} n_j = 0 \text{ or } \boldsymbol{\tau} \cdot \mathbf{T} \cdot \mathbf{n} = 0 \tag{3.42}$$

Equations (3.41) and (3.42) are mathematically equivalent to (3.40) since (3.42) is true for any tangent vector $\boldsymbol{\tau}$. Physically, (3.41) implies that that the normal component of stress across a free surface is balanced by surface tension effects, while (3.42) is a statement that there is no tangential stress.

It is to be noted that (3.40) or its equivalent form (3.41)-(3.42) is only valid when there is fluid motion only on one side of the domain $\partial\Omega$. This is not valid when there is fluid motion on both sides of the interface, as it is for strong winds blowing on top of an ocean surface. Generalization of (3.41) and (3.42) in those cases is that the jump in normal stress across two sides of $\partial\Omega$ is

$$[-p + \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}] = \sigma\kappa$$

where as the jump in tangential stress across two sides of $\partial\Omega$ is

$$[\tau \cdot \mathbf{T} \cdot \mathbf{n}] = 0$$

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(1) Here we are returning to dimensional quantities

Further, unlike the case of a fixed boundary, we need an additional equation to determine the location of the free boundary as part of the problem. This is determined by setting

$$\mathbf{u} \cdot \mathbf{n} = V_n \tag{3.43}$$

where V_n is the normal velocity of the interface. For a free boundary described implicitly by a scalar relation $F(\mathbf{x}, t) = 0$. This relation is found by noting that a point on the free surface can be characterized by $\mathbf{x} = \mathbf{X}(t)$. Then $F(\mathbf{X}(t), t) = 0$; therefore, taking time derivative $F_t + \mathbf{X}_t \cdot (\nabla F) = 0 = F_t + V_n |\nabla F|$, since by definition V_n is the normal component of surface motion, and therefore $\mathbf{X}_t \cdot \mathbf{n} = V_n$. This implies that (3.43) may be replaced by $\mathbf{u} \cdot \mathbf{n} = -\frac{F_t}{|\nabla F|}$, or

$$F_t + \mathbf{u} \cdot (\nabla F) = 0 \ , \ \text{on } F(\mathbf{x}, t) = 0 \tag{3.44}$$

This is called the *kinematic* condition.

3.1. **Inviscid Irrotational Free boundary.** Consider the simplest case, when the flow is inviscid, *i.e.* viscosity effects are neglected. Further, we assume that the flow is irrotational. Then the equations simplify. In Ω, we have

$$\Delta \Phi = 0 \tag{3.45}$$

Then, since Bernoulli equation is valid everywhere in Ω, we have

$$\partial_t \Phi + \frac{p}{\rho} + V(\mathbf{x}) + \frac{1}{2} |\nabla \Phi|^2 = 0, \tag{3.46}$$

where we assumed body force $\mathbf{b} = -\nabla V$. In the case of gravity $V = g x_3$, gravity being alligned in the negative x_3-axis Using this in the pressure equation (3.41), we obtain after noting that viscous stress tension $\mathbf{T} = 0$ here, we obtain on $\partial \Omega$

$$\Phi_t + V(\mathbf{x}) + \frac{1}{2} |\nabla \Phi|^2 = \sigma \kappa \tag{3.47}$$

Equation (3.45) in Ω, together with pressure condition (3.47) and kinematic boundary condition (3.44) completely specifies the free boundary problem if Ω is finite. However, if $\infty \in \Omega$, then we have to add an additional condition at ∞. For instance, a bubble in 3-D with changing volume will introduce a source at ∞ will introduce a source flow at ∞:

$$\mathbf{u} = \nabla \Phi \sim \frac{m(t) \mathbf{x}}{4 \pi |x|^3} \text{ as } \mathbf{x} \to \infty, \tag{3.48}$$

where $m(t) = \frac{d}{dt}(\text{Bubble Volume})$ is source strength. On the other hand, if there is a uniform flow $\mathbf{U}_0$ at ∞, then we need to specify as flow past a solid body,

$$\mathbf{u} \sim \mathbf{U}_0 \text{ as } \mathbf{x} \to \infty \tag{3.49}$$

3.2. **Spherical Bubble.** First consider the simplest case, the motion of spherical oscillating bubble in a fluid with no body force $\mathbf{b} = 0$. This is physically realistic for small bubbles, where gravity does not play an important role. In this case, $\Phi = \Phi(r, t)$, $r = |\mathbf{x}|$. There is no θ, ϕ dependence and the boundary of the sphere is $r = R(t)$ and the domain Ω is given by $r > R(t)$. Therefore, we have from Laplace's equation in spherical coordinates:

$$\partial_r^2 \Phi + \frac{2}{r} \partial_r \Phi = 0 \tag{3.50}$$

This implies

$$\Phi(r, t) = A(t) + \frac{B(t)}{r} \text{ , implying } u_r = -\frac{B(t)}{r^2} \tag{3.51}$$

Therefore, since $F(\mathbf{x}, t) = r - R(t)$, the kinematic condition (3.44) implies that

$$0 = \frac{dR}{dt} - \Phi_r(R(t), t) = \frac{dR}{dt} + \frac{B}{R^2} \tag{3.52}$$

Since increase/decrease in volume of the bubble $\frac{4\pi}{3} R^3(t)$, is effectively a source/sink, it follows that from (3.52),

$$m(t) = \frac{d}{dt} \frac{4\pi}{3} R^3 = 4\pi R^2 \frac{dR}{dt} = -4\pi B \tag{3.53}$$

Now, consider the pressure condition: on $\partial\Omega$, *i.e.* on $r = R(t)$:

$$\partial_t \Phi + \frac{1}{2} (\partial_r \Phi)^2 = \frac{2\sigma}{\rho R} \tag{3.54}$$

From representation (3.51), this becomes

$$\frac{dA}{dt} + \frac{\frac{dB}{dt}}{R} + \frac{B^2}{2R^4} = \frac{2\sigma}{\rho R} \tag{3.55}$$

Further, using Bernoulli equation as $\mathbf{x} \to \infty$, we find from (3.46), (3.55)

$$\frac{p_\infty}{\rho} = -\partial_t \Phi = -\frac{d}{dt} A = +\frac{\frac{dB}{dt}}{R} + \frac{B^2}{2R^4} - \frac{2\sigma}{\rho R} \tag{3.56}$$

Thus, in terms of specified p_∞, by using (3.53) and (3.56), we obtain after some algebra

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{2\sigma}{\rho R} = -\frac{p_\infty}{\rho} \tag{3.57}$$

You can analyze this ODE if you like to study oscillation of bubble for given p_∞/ρ. If p_∞/ρ is a constant (3.57) is a automomous system which can be further reduced to a first order system and further analyzed; or your could study the phase plane the usual way about the equilibrium, which happens to be

$$r = R_0 \equiv -\frac{p_\infty}{2\sigma} \tag{3.58}$$

assuming $p_\infty < 0$. If $p_\infty > 0$, there is no equilibrium, since the bubble will eventually contract to zero size since the acceleration is clearly negative for all time.

You can also study the response of the bubble to sound by considering $\frac{p_\infty}{\rho} = C_0 + C_1 \cos\ \omega t$ for constant C_0 and C_1.

3.3. **Nonspherical Perturbation to a bubble.** Suppose the bubble is now perturbed a bit about the equibrium position $r = R_0$, determined from (3.58) with $p_\infty < 0$ and independent of time and with no body force, *i.e.* $V = 0$. We assume that the perturbation is not necessarily spherically symmetric. In that case, in spherical coordinates, the boundary $\partial\Omega$ of the bubble is given by

$$r = R_0 + \epsilon f(\theta, \phi, t) \tag{3.59}$$

and we seek to study the evolution of f with time, with given initial perturbation

$$f(\theta, \phi, 0) = f_0(\theta, \phi) \tag{3.60}$$

We will choose $\epsilon << 1$, and seek solution for the linearized problem. In order to simplify the problem, we have to take boundary conditions at $r = R_0 + \epsilon f$ and apply them at the spherical unperturbed boundary $r = R_0$. This is done by Taylor expanding the boundary condition in powers of ϵ, assuming that this expansion is possible (this assume *a priori* that the boundary shape is analytic. We note that in equilibrium $\Phi = 0$, since there is no flow in equilibrium. So, we may assume

$$\Phi(r, \theta, \phi, t) = \epsilon \Psi(r, \theta, \phi, t) \tag{3.61}$$

Further, mean curvature is given by $\kappa = \nabla \cdot \mathbf{n}$. In our case, note that in polar coordinates

(3.62)

$$\mathbf{n} = \left(1, -\frac{\epsilon f_\theta}{R_0 + \epsilon f} - \frac{1}{\sin\theta (R_0 + \epsilon f)} f_\phi \right) \left(1 + \frac{\epsilon^2}{(R_0 + \epsilon f)^2} f_\theta^2 + \frac{\epsilon^2 f_\phi^2}{(R_0 + \epsilon f)^2 \sin^2\theta} \right)^{-1/2}$$

Recalling that that for any vector $\mathbf{F}$, expressed in spherical coordinates,

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \partial_r \left(r^2 F_r\right) + \frac{1}{r \sin\theta} \partial_\theta \left(\sin theta F_\theta\right) + \frac{1}{r \sin theta} \partial_\phi F_\phi \tag{3.63}$$

We obtain

$$\kappa = \frac{2}{R_0} - \frac{2\epsilon}{R_0^2} f - \frac{\epsilon^2}{R_0}\mathcal{L}f + O(\epsilon^2) \tag{3.64}$$

where the differential operator $\mathcal{L}$ in θ and ϕ is defined by

$$\mathcal{L} \equiv +\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.65}$$

The pressure boundary condition (3.47) becomes

$$\epsilon\Psi_t(R_0,\theta,\phi) = -\frac{\epsilon\sigma}{R_0^2}(2f + \mathcal{L}f) + O(\epsilon^2) \tag{3.66}$$

The kinematic condition (1.4) becomes

$$\epsilon f_t(\theta,\phi,t) - \epsilon\frac{\partial\Psi}{\partial r}(R_0,\theta,\phi,t) = O(\epsilon^2) \tag{3.67}$$

Equation (3.66) and (3.67) are now applied on the unperturbed boundary $r = R_0$. We have to solve for $\Psi(r,\theta,\phi,t)$ outside this perturbed boundary with condition

$$0 = \Delta\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} \tag{3.68}$$

Now, it is known that the differential operator $\mathcal{L}$ defined in (3.65) has eigenfunctions

$$\mathcal{L}Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi) \tag{3.69}$$

where $l \geq 0$ is an integer, and $Y_{l,m}$ are called spherical harmonics, given by

$$Y_{l,m}(\theta,\phi) = e^{im\phi}P_{l,m}(\cos\theta)\ ,\ \text{where } -l \leq m \leq l \tag{3.70}$$

and $P_{l,m}(z)$ are called associated Legendre functions that satisfy the differential equation

$$\left[(1-z^2)P_{l,m}'\right]' - \frac{m^2}{1-z^2}P_{l,m} = -l(l+1)P_{l,m} \tag{3.71}$$

It is known that the set

$$\{Y_{l,m}(\theta,\phi)\}_{l=0,..\infty,m=-l..l} \tag{3.72}$$

forms a complete orthogonal set on the unit sphere. An arbitrary function $f(\theta,\phi)$ can be written in terms of a linear combination of $Y_{l,m}(\theta,\phi)$ in the $\mathbf{L}^2$ sense. In particular, the we may express perturbation about a sphere is given by

$$f(\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{l,m}(t)Y_{l,m}(\theta,\phi) \tag{3.73}$$

for some set of quantities $a_{l,m}$. Similarly, if we express

$$\Psi(r,\theta,\phi,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{l,m}(r,t) Y_{l,m}(\theta,\phi) \tag{3.74}$$

Plugging (3.74) into (3.68) after using $\mathcal{L} Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}$, we obtain

$$0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}(\theta,\phi) \left(\frac{\partial^2 B_{l,m}}{\partial r^2} + \frac{2}{r}\frac{\partial B_{l,m}}{\partial r} - \frac{l(l+1)B_{l,m}}{r^2} \right), \tag{3.75}$$

implying

$$\frac{\partial^2 B_{l,m}}{\partial r^2} + \frac{2}{r}\frac{\partial B_{l,m}}{\partial r} - \frac{l(l+1)B_{l,m}}{r^2} = 0 \tag{3.76}$$

So, solving we get a linear combination of r^l and r^{-l-1}. The only acceptable solution is a multiple of r^{-l-1} since r^l does not vanish as $r \to \infty$. Therefore, it follows that

$$B_{l,m}(r,t) = \frac{b_{l,m}(t) R_0^{l+1}}{r^{l+1}} \tag{3.77}$$

Using (3.73), (3.74) and (3.77) in the linearized boundary condition (3.66), (3.67), we obtain for each (l,m),

$$\frac{d}{dt} b_{l,m} = -\frac{\sigma}{R_0^2}(-l^2 - l + 2) a_{l,m} \tag{3.78}$$

$$\frac{d}{dt} a_{l,m} + \frac{(l+1)}{R_0} b_{l,m} = 0 \tag{3.79}$$

Or, eliminating $b_{l,m}$ between the two relations, we obtain

$$\frac{d^2 a_{l,m}}{dt^2} = -\frac{(l+1)(l^2+l-2)\sigma}{R_0^3} a_{l,m} \tag{3.80}$$

The solution is obviously sinusoidal for $l \geq 1$, with frequency

$$\omega_l = \sqrt{\frac{(l+1)(l+2)(l-1)\sigma}{R_0^3}} \tag{3.81}$$

This describes the linearized motion of an oscillating bubble that oscillates due to surface tension effects when disturbed from equilibrium The general shape will of course be given by (3.73), with $a_{l,m}$ determined from initial values of shape distortion, that determines $a_{l,m}(0)$ and velocity that determines $b_{l,m}(0)$.

Remark 3.1. *Note that the above calculation involved a linearization and throwing away the nonlinear term. Generally, keeping the nonlinear term makes it a much more difficult mathematical problem. Next class, I will show how you can formulate such free boundary problems in terms of fixed boundary problem in 2-D through the use of conformal map. There is also another way of handling such problems in both 2-D and 3-D. This is through the use of dipole or vortex sheet method.*